Problem 1. A tank initially contains 600 l of salted water whose salt concentration is 0.5 kg/l. Salted water whose salt concentration is 0.25 kg/l flows into the tank at the rate of 12 l/min. The mixture flows out at the same rate. Find the salt content after 20 min.

Let \( S(t) \) be the amount of salt at time \( t \) in kg.

\[
S' = 0.25 \times 12 - \frac{S}{600} \times 12
\]

\[
\therefore S' = 3 - \frac{S}{50}
\]

\[
S' + \frac{S}{50} = 3
\]

We solve using an integrating factor \( u(t) \):

\[
S'u + \frac{S}{50}u = 3u
\]

S.t. \( u' = \frac{u}{50} \) \quad \Rightarrow \quad \ln u = \frac{t}{50} \quad \Rightarrow \quad u = e^{\frac{t}{50}}

\[
\left( e^{\frac{t}{50}} S \right)' = 3e^{\frac{t}{50}}
\]

Integrating both sides yields

\[
e^{\frac{t}{50}} S = 150 e^{\frac{t}{50}} + C
\]

Initial condition: \( S(0) = 0.5 \times 600 = 300 = 150 + C \)

So \( C = 150 \)

Therefore

\[
S = 150 \left( 1 + e^{-\frac{t}{50}} \right)
\]

\[
at t = 20, \quad S(20) = 150 \left( 1 + e^{-\frac{20}{50}} \right)
\]
Problem 2. Find the solution of the differential equation $y'(x) + 2xy(x) = x$, with initial condition $y(1) = 2$.

Use an integrating factor $u(x)$:

$$y'ue^x + 2xye^x = xe^x \quad \text{(6)}$$

Choose $u$ s.t. $u' = 2xu$, i.e. $\frac{u'}{u} = 2x$

$$\Rightarrow u = e^{x^2}$$

The equation \text{(6)} becomes

$$y' e^{x^2} + 2xe^{x^2}y = xe^{x^2}$$

or

$$(ye^{x^2})' = xe^{x^2}$$

Integrating both sides gives

$$ye^{x^2} = \frac{1}{2}e^{x^2} + c$$

$$y(1) = 2 \Rightarrow 2e = \frac{1}{2}e + c \Rightarrow c = 2e - \frac{1}{2}e = \frac{3}{2}e$$

So

$$y = (\frac{1}{2}e^{x^2} + \frac{3}{2}e)e^{-x^2}$$

i.e.

$$y = \frac{1}{2} + \frac{3}{2}e^{-x^2}$$
(5) **Problem 3.** Suppose that $p(\lambda) = \lambda^4(\lambda - 2)(\lambda + 1)^2(\lambda^2 + 16)^2$ is the characteristic polynomial to a certain 12th order (homogeneous) linear differential equation with constant coefficients. What is a fundamental set of solutions?

$$\begin{aligned}
\left\{ & x, x^3, e^x, e^{-x}, x e^{-x}, -x e^{-x}, \\
& \cos 4x, \sin 4x, 8 \sin 4x, 8 \cos 4x \right\}
\end{aligned}$$

\( \lambda = 0 \) (root) \quad \lambda = 2 \quad \lambda = -1 \) (repeated) \quad \lambda = 4i \) (double root)

(5) **Problem 4.** Suppose that a $4 \times 4$-matrix $A$ has the characteristic polynomial $P(\lambda) = (\lambda - 1)^2(\lambda - 3)(\lambda + 5)$. Under what condition is $A$ is diagonalizable?

There are 3 eigenvalues: 1 (double), 3 & 5.

A diagonalizable $\iff \dim \mathcal{E}_1 + \dim \mathcal{E}_3 + \dim \mathcal{E}_5 = 4$

$\iff \dim \mathcal{E}_1 = 2$.

In order for $A$ to be diagonalizable, the eigenspace associated with $\lambda = 1$ must have two linearly independent eigenvectors.
Problem 5. Consider \( L(y) = y''' + 2y'' + 2y' \).

5(a) Solve \( L(y) = 0 \).

Char. poly. \( \text{P}(x) = x^3 + 2x + 2 \Rightarrow x^2 = -2x - 8 = -1 \pm 3i \)

roots are \( x = 0 \) (double) and \( x = -1 \pm 2i \)

A fundamental syst. of soln. is

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} = e^{-x} \begin{bmatrix}
\cos 2x \\
\sin 2x
\end{bmatrix}
\]

5(b) Solve \( L(y) = \cos(x) \Rightarrow \text{Re}[e^{ix}] \)

Since \( i \) is not a root of \( \text{P}(x) \), a (complex) soln. of \( L(y) = e^{ix} \)

is of the form:

\[ y_p = A e^{ix} \]

So \( y_p' = Aie^{ix} \)
\[ y_p'' = -A e^{ix} \]
\[ y_p''' = -Aie^{ix} \]
\[ y_p'''' = A e^{ix} \]

\[ (-1-2i)A = 1 \]
\[ A = \frac{1}{-1-2i} = \frac{1-2i}{5} = \frac{1}{5} + \frac{2}{5}i \]

A part. soln. of \( L(y) = \cos(x) \) is then

\[ y_p = \text{Re}[y_p] = e^x \cos \frac{2}{5} \sin x \]

The general soln. to \( L(y) = \cos(x) \) is:

\[ y = c_1 x + c_2 e^{-x} \cos x + c_3 e^{-x} \sin x + \frac{4}{5} x \cos x - \frac{2}{5} \sin x \]
Problem 6. Consider the following matrix

\( A = \begin{bmatrix} 2 & -1 & 5 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \)

(a) What is the determinant of \( A \)?
\[ \det A = 2 \cdot 2 \cdot 3 \cdot 5 = 60 \]

(b) What is the trace of \( A \)?
\[ \text{tr} A = 2 + 2 + 3 + 5 = 12 \]

(c) What are the eigenvalues of \( A \)?
\[ 2, 3 \text{ and } 5 \]

Problem 7. Consider the matrix

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \]

Find the eigenvalues and eigenvectors of \( A \). Is \( A \) diagonalizable? Justify your answer.

\[ p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 2) \]

Roots are: \( \lambda = 1 \), \( \lambda = -1 \) and \( \lambda = 2 \)

\[ \xi_1: \quad A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \quad x_2 = x_3 = 0 \quad x_1 \text{ arbitrary} \]

\[ \xi_1: \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \xi_1 \]

\[ \xi_1: \quad A + I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad x_3 = 1 \quad \text{(free variable)} \]

\[ x_2 = -2 \quad x_1 = -\sqrt{2} \]

\[ \xi_1: \quad \begin{bmatrix} -1/2 \\ -2 \\ -\sqrt{2} \end{bmatrix} \in \xi_1 \]
\[
\begin{bmatrix}
-1 & 0 \\
0 & -2 & 2
\end{bmatrix}
\]

let \( x_3 = 1 \)

\( x_2 = 1 \)

\( \Rightarrow \)

\( x_1 = 1 \)

So \( v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{E}_2 \).

A is diagonalizable since it has three distinct eigenvalues (and therefore \( \dim \mathcal{E}_1 + \dim \mathcal{E}_2 + \dim \mathcal{E}_3 = 3 \)). In the basis \( \{v_1, v_2, v_3\} \) \( A \) is represented by

\[
D = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

(If \( P = [v_1 \, v_2 \, v_3] \), then \( P^{-1}AP = D \)).

(5) Problem 8. Let \( A \) and \( B \) be two invertible matrices. Show that \( AB \) and \( BA \) have the same characteristic polynomial.

As seen in class, two similar matrices have the same characteristic polynomial. But \( AB \) and \( BA \) are similar since

\[
AB = B^{-1}BAB
\]

* It is easily shown that \( \mathcal{P}_A(d) = \mathcal{P}_{BA}(d) \).

\[
\mathcal{P}_{BA}(d) = \det(PAP - dI) \\
= \det(P^{-1}AP - dP^{-1}P) \\
= \det(P^{-1}(A - dI)P) \\
= \frac{\det P^{-1} \det(A - dI) \det P}{\det P} \\
= \det(A - dI) \\
= \mathcal{P}_A(d).
\]