Assignment 5
(Refer to October 2)

Reading: (from Reed) §6.1, 3.2

Problems:
§2.4: #7, 10
§2.6: #1, 3, 9
§6.1: #1(a,c), 8

Additional Problems:
1. Let \( \{a_n\} \) and \( \{b_n\} \) be Cauchy sequences in an ordered field \( F \). Let \( \{a_n\} \sim \{b_n\} \) mean that \( a_n - b_n \to 0 \). Prove that \( \sim \) is an equivalence relation:
   \( \{a_n\} \sim \{a_n\} \); if \( \{a_n\} \sim \{b_n\} \) then \( \{b_n\} \sim \{a_n\} \); if \( \{a_n\} \sim \{b_n\} \) and \( \{b_n\} \sim \{c_n\} \), then \( \{a_n\} \sim \{c_n\} \).
2. Let \( C(F) \) denote the set of equivalence classes of Cauchy sequences in \( F \). Find an injective function \( F \to C(F) \). (So we can think of \( F \) as a subset of \( C(F) \), \( F \subseteq C(F) \): we have “enlarged” \( F \).)
3. Prove that the sum and product of Cauchy sequences is Cauchy.
4. Let \([a_n]\) denote the equivalence class containing the Cauchy sequence \( \{a_n\} \). Given Cauchy sequences \( \{a_n\} \) and \( \{b_n\} \), define the sum and product of the equivalence classes containing them by
   \[
   [a_n] +_{C(F)} [b_n] := [a_n + b_n]
   \]
   \[
   [a_n] \cdot_{C(F)} [b_n] := [a_n b_n]
   \]
   Prove that these rules are well-defined by showing that if \( \{a_n\} \sim \{a'_n\} \) and \( \{b_n\} \sim \{b'_n\} \), then \( \{a_n + b_n\} \sim \{a'_n + b'_n\} \) and \( \{a_n b_n\} \sim \{a'_n b'_n\} \).
5. If \( C(F) \) denotes the set of equivalence classes of Cauchy sequences in \( F \), then with the sum and product operations in 3, \( C \) is in fact a field in such a way that the “copy” of \( F \) in \( C(F) \) in 2. above is the field \( F \) we started with: \( F \subseteq C(F) \) is a subfield. Don’t try to prove this, but identify the additive and multiplicative identities 0 and 1 in \( C(F) \) and verify that \( [a_n] +_{C(F)} 0 = [a_n] \) and \( [a_n] \cdot_{C(F)} 1 = [a_n] \) for all Cauchy sequences \( \{a_n\} \). (Keep in mind that your choice of 0 (or 1) in your answer will be an equivalence class of Cauchy sequences. This class may be identified by specifying any Cauchy sequence in it.)