Assignment 10
(Due November 25)

Reading: (from Reed) §5.3
Problems: §4.3: #7
§5.1: #1, 4, 11
§5.2: #3, 6, 9

Additional Problems:
1. Let \( f(x) \) be \( n + 1 \) times continuously differentiable on an open interval \( I \) containing \( a \in \mathbb{R} \). Use integration by parts to show that the (integral form of the) Taylor remainder for \( f(x) \)
\[
R_n(x; a) := \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt
\]
satisfies the equation
\[
R_n(x; a) = R_{n-1}(x; a) - \frac{f^{(n)}(a)}{n!} (x-a)^n.
\]
on \( I \). Use this to prove Taylor’s formula: for any \( n \geq 1 \) and \( x \in I \)
\[
f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i + R_n(x; a).
\]

2. Use Taylor’s theorem to show that if \( f \) and \( g \) are \( (n+1) \)-times continuously differentiable functions on an open interval containing \( a \), \( f^{(k)}(a) = g^{(k)}(a) = 0 \) for \( k = 0, 1, \ldots, n \), and \( g^{(n+1)}(a) \neq 0 \), then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n+1)}(a)}{g^{(n+1)}(a)}
\]

3. Use 2. to compute
\[
\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}
\]

4. Let \( \{f_n\} \) be a sequence of functions which converges pointwise to a function \( f \) on some subset \( E \subseteq \mathbb{R} \). Suppose there exists a sequence \( \{x_n\} \subseteq E \) and a positive number \( c \) such that \( |f_n(x_n) - f(x_n)| > c \), for all \( n \). Prove that \( \{f_n\} \) does not converge uniformly to \( f \) on \( E \).

5. Let \( \{r_1, r_2, \ldots\} \) be the set of rational numbers in \( [0, 1] \). For \( x \in [0, 1] \) and \( n \in \mathbb{N} \), let
\[
f_n(x) = \begin{cases} 
1, & x = r_1, \ldots, r_n \\
0, & \text{otherwise}
\end{cases}
\]
and
\[
f(x) = \begin{cases} 
1, & x \text{ rational} \\
0, & x \text{ irrational}
\end{cases}
\]
Prove that \( f_n \to f \) pointwise but not uniformly. (Note that \( f_n \) is Riemann integrable but \( f \) is not.)