1. Let $H$ be a subgroup of finite index in a group $G$, and let $\text{Perm}(G/H)$ denote the group of permutations of the set $G/H$. (If $[G : H] = n$ and we choose a bijection between the set $G/H$ and the set $\{1, ..., n\}$, then $\text{Perm}(G/H)$ is identified with $S_n$.) For each $g \in G$, define $p_g \in \text{Perm}(G/H)$ by $p_g(g'H) = gg'H$. Prove that this rule defines a homomorphism

$$P : G \to \text{Perm}(G/H)$$

such that $\ker P < H$.

2. Use 1. to prove that if $p$ is prime, then every group of order $p^2$ has a normal subgroup of order $p$.

3. Let $n$ be an integer $> 1$ and let $\gamma : (\mathbb{Z}/n\mathbb{Z})^\times \to \text{Aut}(\mathbb{Z}/n\mathbb{Z})$ be the function defined by

$$\gamma(\bar{k}) := \{ \bar{a} \mapsto \bar{k}\bar{a} \}$$

(Here $\text{Aut}(\mathbb{Z}/n\mathbb{Z})$ denotes the group of isomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, where the group operation is composition.) Prove that $\gamma$ is an isomorphism of groups.

4. Use 2. and 3. to prove that all groups of order $p^2$ are abelian. (Suggestion: Let $H < G$ be normal and of order $p$, and pick $g \in G - H$ which you can assume (why?) to be of order $p$. Define a homomorphism $< g > \to \text{Aut} H$ by sending $g^n$ to $\{ h \mapsto g^n hg^{-n} \}$. Show this homomorphism is trivial and conclude that $G$ is abelian.)

5. Let $G = HK$, where $H$ and $K$ are subgroups of the group $G$ (not necessarily finite). Prove that if $H \cap K = \{e\}$, then each element $g \in G$ can be written $g = hk$, for unique $h \in H$ and $k \in K$. 