Assignment 11  
(Due November 18)

Reading: §8.1, 9.1, 9.2, 10.1, 10.2

Problems

1. Let $D$ be a Euclidean domain, and let $a, b \in D$. Use the Euclidean Algorithm in $D$ to prove that there is an element $d \in D$ such that the principal ideal $(d) = (a, b)$, the ideal generated by $a$ and $b$. (See Additional Problem 1. in Assignment 1.)

2. By #8, §7.4, the element $d$ in 1. is unique up to multiplication by a unit of $D$. We say $a$ and $b$ are relatively prime if $d$ is itself a unit. Prove that $a$ and $b$ are relatively prime if and only if whenever $c | a$ and $c | b$, $c$ is a unit. Prove that if $a | bc$ in $D$ and $a$ and $b$ are relatively prime, then $a | c$.

3. In general there is no good notion of “greatest common divisor” in $D$, at least if one wants the notion of “greatest” to come from an ordering of $D$. (For example, $\mathbb{Z}[i]$ has no good notion of ”positive number”.) Show however that there is a canonical choice for the element $d$ in 1. above when $D = F[x]$, where $F$ is a field. We then call this $d$ the greatest common divisor of $a$ and $b$.

4. Let $R$ be a ring-with-identity. A subset $\{e_1, \ldots, e_n\} \subseteq C(R)$ (the center of $R$) is called a complete set of central, orthogonal idempotents if $\sum e_i = 1$, $e_i^2 = e_i$ for all $i$ and $e_i e_j = 0$ for all $i \neq j$. Prove that there is an isomorphism of rings

$$R \cong Re_1 \times \cdots \times Re_n, \ r \mapsto (re_1, \ldots, re_n)$$

and that for each $i$ there is an isomorphism of rings

$$Re_i \cong R/(1 - e_i), \ re_i \mapsto rei$$

5. Use the first part of 4. to prove that if $a, b \in D$, a Euclidean domain, and if $a$ and $b$ are relatively prime, then

$$D/(ab) \cong D/(a) \times D/(b)$$

6. Let $H$ be a normal subgroup of a group $G$. The subgroup correspondence theorem (Theorem 20, p. 99 in the text) states that there is a 1-1 order-preserving correspondence between the subgroups $K$ of $G$ containing $H$ and the subgroups $\overline{K}$ of $G/H$, given by $K \mapsto \pi(K) = K/H$, where $\pi : G \to G/H$ is the canonical homomorphism. State and prove the analogue for rings and ideals.

7. Let $H$ be a normal subgroup of a group $G$. The universal property of $\pi : G \to G/H$ (bottom of p.100 in the text) states that if $\phi : G \to K$ is a homomorphism of groups such
that $H \leq \ker \phi$, then there is a unique homomorphism $\bar{\phi} : G/H \to K$ such that $\bar{\phi} \circ \pi = \phi$. State and prove the analogue for rings and ideals.

8. State and prove the analogues of Theorem 16, p 97 and Theorem 17, p. 98 in the text for rings and ideals.

9. Let $X$ be a subset of $\mathbb{R}^3$ (more generally, $X$ could be any topological space), let $C(X)$ denote the ring of continuous functions $X \to \mathbb{R}$ and let $Z \subseteq X$. Prove that $I(Z) := \{ f \in C(X) | f(Z) = 0 \}$ is an ideal in $C(X)$ which maximal if $Z$ is a single point.