Math 121 March 24, 2005

Assignment 9
(Due Thursday, March 31, 2005)

Reading: §4.3

Problems: §4.2: #1, 3(a,b,c), 6, 11(a), 20, 23(a)

Additional Problems: Let R be a commutative ring. Let r_1, r_2 be elements of R. We say r_1 and r_2 are *associates* if there is a unit u in R such that $r_1 = ur_2$.

1. What integers n_1 and n_2 are associates in $\mathbb{Z}$? What $f(x)$ and $g(x)$ in $F[x]$ are associates?

   We say that *UF holds in R* if

   i. (Existence) Every non-unit $a \in R$ may be expressed as a finite product of irreducibles:
   $$a = i_1 i_2 \dots i_m$$
   and

   ii. (Uniqueness) If
   $$i_1 i_2 \dots i_m = j_1 j_2 \dots j_n$$
   are products of irreducibles, then $m = n$ and, re-ordering if necessary, i_l is an associate of j_l, $l = 1, 2, \dots, m$.

2. Show that UF (in the above sense) holds in $\mathbb{Z}$ and $F[x]$.

3. Suppose that UF holds in R. Suppose that the irreducible i divides the non-unit a. Show that there is an integer $N \geq 2$ such that i^n does not divide a, for all $n \geq N$. (In other words, there is a highest power of i dividing a.)

4. By 2., UF holds in the rings $\mathbb{Z}_2[x]$ and $\mathbb{Z}_3[x]$. Show however that UF does not hold in the product $\mathbb{Z}_2[x] \times \mathbb{Z}_3[x]$ of these rings. (*Suggestion:* According to Assignment 8, Additional Problem 3, $(1, q(x)) \in \mathbb{Z}_2[x] \times \mathbb{Z}_3[x]$ is irreducible in $\mathbb{Z}_2[x] \times \mathbb{Z}_3[x]$, provided $q(x)$ is irreducible in $\mathbb{Z}_3[x]$. Notice that for any $p(x) \in \mathbb{Z}_2[x]$,
$$(p(x), 0) = (1, q(x))^k (p(x), 0)$$
for all $k \geq 1$. Apply 3.)

Comment: The fact that $\mathbb{Z}_2[x] \times \mathbb{Z}_3[x]$ is not an integral domain seems to be causing the failure of UF. Or has UF failed because we tried to factor a zero-divisor into irreducibles? To be continued...