On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization.

William K. Allard
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Thanx to Kevin Vixie who introduced me to these problems while I was visiting Los Alamos National Laboratory last year.
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In case \( n > 7 \) problems arise because our work will depend in an essential way on the regularity of area minimizing hypersurfaces in \( \mathbb{R}^n \).

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Let \( s \) be a "noisy" grayscale image. For example, let \( s: \mathbb{R}^n \to [0, 1] \) be measurable with bounded support.

We seek to replace \( s \) with \( f: \mathbb{R}^n \to \mathbb{R} \) which is not "noisy".

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How far is $f$ from $s$?

Suppose $\beta : [0, \infty) \rightarrow [0, \infty)$ is increasing and zero at zero and $\gamma : \mathbb{R} \rightarrow [0, \infty)$ is even, increasing and zero at zero.

For any real valued measurable function $f$ on $\mathbb{R}^n$ let $w(f) = \beta(\int \gamma(f(x) - s(x)) \, dL_nx)$ be the "distance" from $f$ to $s$.

Assume $w(s) < \infty$. On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization. – p.5/27
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For example, Rudin, Osher and Fatemi (ROF) study
\[ \beta(y) = y, \quad y \in [0, \infty) \]
and
\[ \gamma(y) = \frac{1}{2} y^2, \quad y \in \mathbb{R} \]
so
\[ w(f) = \frac{1}{2} \int |f - s|^2 dL. \]
Chan and Esedoglu (CE) use the same \( \beta \) but let
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It may not make much difference, as we shall see later.

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For example, Rudin, Osher and Fatemi (ROF) study \( \beta(y) = y, y \in [0, \infty) \) and \( \gamma(y) = \frac{1}{2}y^2, y \in \mathbb{R} \) so \( w(f) = \frac{1}{2} \int |f - s|^2 dL \).

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It may not make much difference, as we shall see later.
Total variation regularization

But we need to "regularize" \( f \). So let \( \epsilon > 0 \) and require that \( f \) be a minimizer of

\[
W_\epsilon(f) = \epsilon \text{TV}(f) + \beta \int_\mathbb{R}^n \gamma(f(x) - s(x)) \, d\mu(x);
\]

where \( \text{TV}(f) = \sup \{ \| \int f \, \text{div} X \, d\mu \| : X \in X(\mathbb{R}^n) \} \) and \( X(\mathbb{R}^n) \) is the space of smooth compactly supported vector fields on \( \mathbb{R}^n \).
Some basic facts.

Suppose $f$ is a minimizer for $W \in \mathbb{R}$. Then $\text{ess sup} |f| \leq \text{ess sup} |s|$ and the support of $f$ is contained in the convex hull of the support of $s$. 

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Suppose $f$ is a minimizer for $W_\epsilon$. Then $\text{ess sup} |f| \leq \text{ess sup} |s|$ and the support of $f$ is contained in the convex hull of the support of $s$. 

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$D_m(\mathbb{R}^n)$ is the space of smooth compactly supported differential $m$-forms on $\mathbb{R}^n$ with the strong topology. Its dual, denoted by $D_m(\mathbb{R}^n)$, is the space of $m$-dimensional currents in $\mathbb{R}^n$.

Let $||T||$ be the largest Borel regular measure on $\mathbb{R}^n$ such that $||T||(G) = \sup\{||T(\omega)|| : \omega \in D_m(\mathbb{R}^n) \text{ and } ||\omega|| \leq 1\}$.

We say $T$ is representable by integration if $||T||$ is a Radon measure.

Suppose $T \in D_m(\mathbb{R}^n)$. Its boundary $\partial T$, defined by setting $\partial T(\omega) = T(d\omega)$, $\omega \in D_{m-1}(\mathbb{R}^n)$, is evidently in $D_{m-1}(\mathbb{R}^n)$.

This is making Stokes' Theorem into a definition.
$D^m(R^n)$ is the space of smooth compactly supported differential $m$-forms on $R^n$ with the strong topology. Its dual, denoted by $D^m(R^n)$, is the space of $m$-dimensional currents in $R^n$.

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$\mathcal{D}^m(n)$ is the space of smooth compactly supported differential $m$-forms on $\mathbb{R}^n$ with the strong topology. Its dual, denoted by $\mathcal{D}^m(\mathbb{R}^n)$, is the space of $m$-dimensional currents in $\mathbb{R}^n$. Let $||T||$ be the largest Borel regular measure on $\mathbb{R}^n$ such that $||T||(G) = \sup\{||T(\omega)|| : \omega \in \mathcal{D}^m(\mathbb{R}^n) \text{ and } ||\omega|| \leq 1\}$.

We say $T$ is representable by integration if $||T||$ is a Radon measure. Suppose $T \in \mathcal{D}^m(\mathbb{R}^n)$. Its boundary $\partial T$, defined by setting $\partial T(\omega) = T(d\omega)$, $\omega \in \mathcal{D}^{m-1}(\mathbb{R}^n)$, is evidently in $\mathcal{D}^{m-1}(\mathbb{R}^n)$. This is making Stokes' Theorem into a definition.
Suppose $f \in L_{\text{loc}}^1(\mathbb{R}^n)$. Let $[f] \in D_n(\mathbb{R}^n)$ be such that $[f](\omega) = \int f(x) \omega(x)(e_1, \ldots, e_n) \, dL_n^x$ for $\omega \in D_n(\mathbb{R}^n)$.

Then $\text{TV}(f) = ||\partial [f]||(\mathbb{R}^n)$.

If $f \geq 0$ we have the stacking formulae $[f] = \int_0^\infty [\{f \geq y\}] \, dL_1^y$ and $\partial [f] = \int_0^\infty \partial [\{f \geq y\}] \, dL_1^y$; $||\partial [f]|| = \int_0^\infty ||\partial [\{f \geq y\}]|| \, dL_1^y$;
Suppose \( f \in L^{1}_{\text{loc}}(\mathbb{R}^{n}) \). Let \([f] \in D_{n}(\mathbb{R}^{n})\) be such that 

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[f](\omega) = \int f(x) \omega(x)(e_{1}, \ldots, e_{n}) \, d\mathbb{L}^{n}(x)
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Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $[f] \in \mathcal{D}^n(\mathbb{R}^n)$ be such that

$$[f](\omega) = \int f(x) \omega(x)(e_1, \ldots, e_n) dL^n_x$$

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Suppose $f \in L_{\text{loc}}^1(\mathbb{R}^n)$. Let $[f] \in \mathcal{D}_n(\mathbb{R}^n)$ be such that $[f](\omega) = \int f(x) \omega(x)(e_1, \ldots, e_n) \, d\mathbb{L}^n_x$ for $\omega \in \mathcal{D}_n(\mathbb{R}^n)$. Then $\text{TV}(f) = ||\partial [f]||(\mathbb{R}^n)$.

If $f \geq 0$ we have the stacking formulae $[f] = \int_0^\infty \{ f \geq y \} \, d\mathbb{L}^1_y$ and $\partial [f] = \int_0^\infty \partial \{ f \geq y \} \, d\mathbb{L}^1_y$; $||\partial [f]|| = \int_0^\infty ||\partial \{ f \geq y \}|| \, d\mathbb{L}^1_y$.
The space \( \lambda(\mathbb{R}^n) \) consists of those \( f \in \text{BV}_{\text{loc}}(\mathbb{R}^n) \) such that

\[
\|\partial[f]\|_K \leq \|\partial[g]\|_K + \lambda \int |f - g| dL_n
\]

whenever \( g \in \text{BV}_{\text{loc}}(\mathbb{R}^n) \), \( K \) is a compact subset of \( \mathbb{R}^n \), and \( \text{spt}[f - g] \subset K \) and \( \text{ess inf} f \leq g \leq \text{ess sup} f \).

**Theorem.** A minimizer for the (CE) model is in \( B^1/\epsilon(\mathbb{R}^n) \). A similar result holds for the model in general as well as for models which are Lipschitz continuous in the \( L_1 \) norm.
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Theorem. A minimizer for the (CE) model is in $\mathcal{B}_{1/\epsilon}(\mathbb{R}^n)$. A similar result holds for the model in general as well as for models which are Lipschitz continuous in the $L^1$ norm.

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The space $\lambda(R^n)$ consists of those $f \in BV_{\text{loc}}(R^n)$ such that $\|\partial f\|_K \leq \|\partial g\|_K + \lambda \int |f-g| \, dL^n$ whenever $g \in BV_{\text{loc}}(R^n)$, $K$ is a compact subset of $R^n$ and $\text{spt}[f-g] \subset K$ and $\text{ess inf} f \leq g \leq \text{ess sup} f$.

Theorem. A minimizer for the (CE) model is in $B_{1/\epsilon}(R^n)$. A similar result holds for the model in general as well as for models which are Lipschitz continuous in the $L^1$ norm.
A simple proof.

Suppose $f$ is a minimizer for the CE model. Let $g$ be such that the support of $g - f$ is compact.

Then

$$\epsilon \text{TV}(f) + \int |f - s| = W \epsilon(f) \leq W \epsilon(g) = \epsilon \text{TV}(g) + \int |g - s| - |f - s| \leq \epsilon \text{TV}(g) + 1 \epsilon \int |f - g|.$$
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Suppose $f$ is a minimizer for the CE model. Let $g$ be such that the support of $g - f$ is compact. Then

$$\epsilon_{TV}(f) + \int |f - s| = \epsilon_{TV}(g) + \int |g - s| \leq \epsilon_{TV}(g) + 1 \int |g - s| - |f - s| \leq \epsilon_{TV}(g) + 1 \epsilon_{TV}(f).$$

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A simple proof.

Suppose \( f \) is a minimizer for the CE model. Let \( g \) be such that the support of \( g - f \) is compact. Then

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\epsilon TV(f) + \int |f - s| = W\epsilon(f) \leq W\epsilon(g) = \epsilon TV(g) + \frac{1}{\epsilon} \int |g - s| - |f - s| \leq \epsilon TV(g) + \frac{1}{\epsilon} \int |f - g|.
\]

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Regularity Theorem for $B_{\lambda}(\mathbb{R}^n)$

For each $\lambda > 0$ there are $C: (0, 1) \to (0, \infty)$ and $\rho > 0$ such that if $f \in B_{\lambda}(\mathbb{R}^n)$ and $y \in \mathbb{R}$ then $E = \text{spt}(\partial\{f \geq y\})$ is an embedded $C^1$ submanifold of $\mathbb{R}^n$ (possibly empty) with the property that if $N$ is a continuous field of normals to $E$ then

$$|N(x) - N(a)| \leq C|x - a|$$

whenever $x, a \in E$ and $|a - b| \geq \rho$ if $a, b \in E$, $a \neq b$ and $N(a) = \pm N(b)$.

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Theorem. Suppose $f \in B_{\lambda}(\mathbb{R}^n)$ and $y \in \mathbb{R}$. Then $tf, f + t, f \land t$ and $f \lor t$ are in $B_{\lambda}(\mathbb{R}^n)$.

Theorem. Suppose $f \in L_{loc}^{1}(\mathbb{R}^n)$. The following are equivalent:

$f \in B_{\lambda}(\mathbb{R}^n) \iff \{f \geq y\} \in B_{\lambda}(\mathbb{R}^n)$ for all $y \in \mathbb{R}$.
Theorem. Suppose $f \in \mathcal{B}_\lambda (\mathbb{R}^n)$ and $y \in \mathbb{R}$. Then $tf$, $f + t$, $f \land t$ and $f \lor t$ are in $\mathcal{B}_\lambda (\mathbb{R}^n)$.

Theorem. Suppose $f \in L_{loc}^1 (\mathbb{R}^n)$. The following are equivalent:

1. $f \in \mathcal{B}_\lambda (\mathbb{R}^n)$
2. $\{ f \geq y \} \in \mathcal{B}_\lambda (\mathbb{R}^n)$ for all $y \in \mathbb{R}$.

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Some assumptions.

Let's assume $\beta(y) = y$ for $y \in [0, \infty)$; it really doesn't make that much difference.

Let us also assume that $U$ is an open subset of $\mathbb{R}^n$, $s$ is smooth on $U$, $\gamma$ is convex and $J$ is an open interval such that $U \times J \ni (x, y) \mapsto \gamma(y - s(x))$ is smooth.

In the (CE) model this last condition may be replaced by the condition that either $s \geq f$ or $f \leq s$ essentially on $U$.

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In the (CE) model this last condition may be replaced by the condition that either $s \geq f$ or $f \leq s$ essentially on $U$. On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization.
Precise mean curvature information.

Theorem. Suppose $f$ is a minimizer of $W_{\epsilon}$, $y \in J$, and $E = U \cap \text{spt} \partial \{ f \geq y \} \neq \emptyset$.

Then $E$ is smooth.

Moreover, if $\Pi$ is the second fundamental form of $E$ relative to $N = n \{| f \geq y \}|E$ then $\text{trace} \Pi(x) = \frac{1}{\epsilon} \gamma'(y - s(x))$, $x \in E$.

By the way, $\Pi$ is defined in such a way that if $\{ f \geq y \} = \{| x | \leq R \}$ for some positive $R$ then $\text{trace} \Pi(x) = -n - 1 R$. 

On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization. – p.17/27
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Then $E$ is smooth. Moreover, if $\Pi$ is the second fundamental form of $E$ relative to $N = n\{|f \geq y\}|E$ then $\text{trace}\,\Pi(x) = 1/\epsilon \gamma'(y - s(x))$, $x \in E$.

By the way .., $\Pi$ is defined in such a way that if $\{f \geq y\} = \{|x| \leq R\}$ for some positive $R$ then $\text{trace}\,\Pi(x) = -n^{-1}R$. 

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Precise mean curvature information.

Theorem.

Suppose \( f \) is a minimizer of \( W_{\epsilon} \), \( y \in J \) and \( E = U \cap \text{spt} \partial \{ f \geq y \} \neq \emptyset \).

Then \( E \) is smooth.

Moreover, if \( \Pi \) is the second fundamental form of \( E \) relative to \( N = n \{ f \geq y \} | E \) then

\[
\text{trace } \Pi(x) = \frac{1}{\epsilon} \gamma' (y - s(x)), x \in E.
\]

By the way .., \( \Pi \) is defined in such a way that if \( \{ f \geq y \} = \{|x| \leq R\} \) for some positive \( R \) then

\[
\text{trace } \Pi(x) = -n - 1 R.
\]
Precise mean curvature information.

Theorem. Suppose $f$ is a minimizer of $W_{\epsilon}$, $y \in J$ and $E = U \cap \text{spt} \partial \{f \geq y\} \neq \emptyset$.

Then $E$ is smooth. Moreover, if $\Pi$ is the second fundamental form of $E$ relative to $N = \frac{1}{\text{spt} \partial \{f \geq y\}}|E|$, then

$$\text{trace } \Pi(x) = \frac{1}{\epsilon} \gamma'(y - s(x)), \quad x \in E.$$ 

By the way, $\Pi$ is defined in such a way that if $\{f \geq y\} = \{|x| \leq R\}$ for some positive $R$, then

$$\text{trace } \Pi(x) = -n - 1 \frac{R}{\epsilon}.$$
Precise mean curvature information.

Theorem. Suppose $f$ is a minimizer of $W_{\epsilon}$, $y \in J$ and $E = U \cap \text{spt} \partial \{ f \geq y \} \neq \emptyset$.

Then $E$ is smooth. Moreover, if $\Pi$ is the second fundamental form of $E$ relative to $N = n |_{\{ f \geq y \} \setminus E}$ then

$$\text{trace} \, \Pi(x) = \frac{1}{\epsilon} \gamma'(y - s(x)), \quad x \in E.$$ 

By the way, $\Pi$ is defined in such a way that if $\{ f \geq y \} = \{|x| \leq R\}$ for some positive $R$, then

$$\text{trace} \, \Pi(x) = -n - 1 R.$$ 

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The second variation inequality. 

Theorem. 

Same assumptions and definitions of previous Theorem. 

\[
\int_{E} |\partial \psi|^2 dH^{n-1} \geq \int_{E} \psi^2 \left(\gamma'(y-s)^2 - \epsilon^2 \gamma''(y-s) \nabla s \cdot N\right) dH^{n-1}.
\]

for any smooth real valued compactly supported function \(\psi\) on \(E\).
The case \( n = 2 \).

Suppose \( P : [0, L] \to E \) is an arclength parameterization of a connected component of \( E \). Then the second variation inequality says that

\[
\int_{L}^{0} (\zeta')^2 \geq \int_{L}^{0} \zeta^2 ( (\gamma' (y - s \circ P)) \epsilon^2 - \gamma'' (y - s \circ P) \nabla s \cdot N \epsilon^2 )
\]

whenever \( \zeta \) is \( C^1 \) on \( [0, L] \) and vanishes at 0 and L.

Suppose \( s \) is constant on \( U \). Let \( \zeta (\sigma) = \sin (\pi \sigma / L) \), \( \sigma \in [0, L] \).

infer that if \( R \) is the radius of curvature of the range of \( P \) then \( L \leq \pi R \) for the (CE) model or for the (ROF) model.

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The case \( n = 2 \).

Suppose \( P : [0, L] \to E \) is an arclength parameterization of a connected component of \( E \). Then the second variation inequality says that

\[
\int_0^L (\zeta'')^2 \geq \int_0^L \zeta^2 \left( (\gamma' (y - s \circ P) \epsilon)^2 - \gamma'' (y - s \circ P) \nabla s \cdot N \epsilon \right)
\]

whenever \( \zeta \) is \( C^1 \) on \( [0, L] \) and vanishes at 0 and \( L \).

Suppose \( s \) is constant on \( U \). Let \( \zeta (\sigma) = \sin (\pi \sigma / L) \), \( \sigma \in [0, L] \).

infer that if \( R \) is the radius of curvature of the range of \( P \) then

\( L \leq \pi R \) for the (CE) model or for the (ROF) model.
We're in $\mathbb{R} \times \mathbb{R}$; the blue stuff is $\{ (x, y) : 0 \leq f(x) \leq y \}$.

If the purple line is at $x = a$ then the length of the solid part is $f(a)$.

Blue plus red when sliced, say where the green is, will define a comparison to $f$, call it $g$. If the green line is at $x = b$ then the sum of the lengths of the green solid pieces is $g(b)$. On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization.
A simple case which we can solve.

Theorem. Suppose \( n = 2 \), \( s \) is the indicator function of a closed convex set \( S \), \( f \) is a minimizer of \( W \epsilon \), \( 0 \leq y \leq 1 \) and \( E = \text{spt} \left( \{ f \geq y \} \right) \neq \emptyset \). Then \( E \) is the union of the closed balls of radius of \( \epsilon \) which are contained in \( S \).
A simple case which we can solve.

**Theorem.** Suppose $n = 2$, $s$ is the indicator function of a closed convex set $S$, $f$ is a minimizer of $W \in [0, 1]$, and $E = \text{spt}(\{f \geq y\}) \neq \emptyset$.

Then $E$ is the union of the closed balls of radius of $\epsilon$ which are contained in $S$. On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization. – p.21/27
A simple case which we can solve.

Theorem. Suppose $n = 2$, $s$ is the indicator function of a closed convex set $S$, $f$ is a minimizer of $W_\epsilon$, $0 \leq y \leq 1$ and $E = \text{spt}(\{f \geq y\}) \neq \emptyset$.

Then $E$ is the union of the closed balls of radius of $\epsilon$ which are contained in $S$. 

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A simple case which we can solve.

**Theorem.**

Suppose $n = 2$, $s$ is the indicator function of a closed convex set $S$, $f$ is a minimizer of $W_{\epsilon}$, $0 \leq y \leq 1$ and $E = \text{spt}(\{f \geq y\}) \neq \emptyset$.

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Theorem. Suppose $n = 2$, $s$ is the indicator function of a closed convex set $S$, $f$ is a minimizer of $W_{\epsilon}$, $0 \leq y \leq 1$ and $E = \text{spt}(\{ f \geq y \}) \neq \emptyset$. Then $E$ is the union of the closed balls of radius of $\epsilon$ which are contained in $S$.

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An example for the CE model

The blue curve is the boundary of $S$ which has area considerably larger than $\epsilon^2$. 

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Still another example for the CE model.

The vertical separation between the blue rectangles must be less than $2\epsilon$. 

On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization.
The answer is...
An example for the ROF model

Suppose $s$ is the indicator function of a square of side length $2$. Let's look near the upper left hand corner.

Remember:

$$R = 1 - y \epsilon, \quad y = 1 - \epsilon \quad R, \quad \epsilon < R < 1.$$
Another example for the ROF model

Suppose \( s \) is the indicator function of a something whose upper left hand corner is in blue. Remember:

\[
R = \begin{cases} 
\epsilon - y & \text{on the inside}, \\
\epsilon y & \text{on the outside}.
\end{cases}
\]

On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization.