Abstract

Let $\Omega$ be an open subset of $\mathbb{R}^n$ where $2 \leq n \leq 7$; the reason for restriction on $n$ is that our work will make use of the regularity theory for area minimizing hypersurfaces.

Let $s \in L_\infty(\Omega)$, let $\gamma : [0, \infty) \to [0, \infty)$ be zero at zero, nondecreasing and convex and for $f \in L_\infty(\Omega)$ let

$$F(f) = \int_\Omega \gamma(|f(x) - s(x)|) \, d\mathcal{L}^2 x;$$

$\mathcal{L}^2$ here is Lebesgue measure on $\mathbb{R}^n$. In the denoising literature $F$ would be called a fidelity term in that it measures deviation from $s$ which could be a noisy grayscale image. Let $\epsilon > 0$ and, for $f \in L_\infty(\Omega)$, let

$$F_\epsilon(f) = \epsilon \text{TV}(f) + F(f);$$

here $\text{TV}(f)$ is the total variation of $f$. A minimizer of $F_\epsilon$ is called a total variation regularization of $s$. Rudin, Osher and Fatemi and Chan and Esedoglu have studied total variation regularizations of $F$ where $\gamma(y) = y^2$ and $\gamma(y) = y$, $y \in [0, \infty)$, respectively.

Let $f$ be a total variation regularization of $F$. The first main result of this paper is that the reduced boundaries of the sets $\{f \geq y\}$, $y \in \mathbb{R}$ are $C^{1+\mu}$ hypersurfaces for any $\mu \in (0, 1)$ in case $n > 2$ and any $\mu \in (0, 1]$ in case $n = 2$; moreover, the generalized mean curvature of the sets $\{f \geq y\}$ will be bounded by constants one can readily determine from the norm of $s$ in $L_\infty(\Omega)$. In fact, this result holds for a rather general class of fidelities.

A second result gives precise curvature information about the reduced boundary of $\{f \geq y\}$ near points where $s$ is smooth. This curvature information will allow us to construct a number of interesting examples of total variation regularizations.
On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization; II. Examples.

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1 Introduction.

1.1 Total variation regularization.

Let

\[ \mathcal{F}(\mathbb{R}^2) = \{ f \in L_1(\mathbb{R}^2) \cap L_\infty(\mathbb{R}^2) : f \geq 0 \}. \]

Suppose

(i) \( s \in \mathcal{F}(\mathbb{R}^2) \);
(ii) \( \gamma : \mathbb{R} \to [0, \infty), \gamma \) is convex, \( \gamma(0) = 0 \) and \( \gamma(y) > 0 \) if \( y \in \mathbb{R} \sim \{0\} \);
(iii) \( F(f) = \int_{\mathbb{R}^2} \gamma(f(x) - s(x)) \, d\mathcal{L}^2 \, x \) for \( f \in \mathcal{F}(\mathbb{R}^2) \);

here \( \mathcal{L}^2 \) is Lebesgue measure on \( \mathbb{R}^2 \). Thus \( F(f) \) is a measure of how much \( f \) differs from \( s \).

Whenever \( f \in L_1^{loc}(\mathbb{R}^2) \) we let \( TV(f) \), the total variation of \( f \), be the supremum of

\[ \int_{\mathbb{R}^2} f \text{div} X \, d\mathcal{L}^2 \]

corresponding to smooth compactly supported vector fields \( X \) on \( \mathbb{R}^2 \) such that \( |X(x)| \leq 1 \) for \( x \in \mathbb{R}^2 \); note that \( TV(f) \) here would be \( TV(f, \mathbb{R}^2) \) in [AW]. As is well known, if \( f \) is \( C^1 \) on \( \mathbb{R}^2 \) then

\[ TV(f) = \int_{\mathbb{R}^2} |\nabla f| \, d\mathcal{L}^2 \]

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and if $E$ a subset of $\mathbb{R}^2$ with Lipschitz boundary and with indicator function $1_E$ then $\text{TV}(1_E)$ equals the length of the boundary of $E$. We say $f$ is of bounded variation (on $\mathbb{R}^2$) if $f \in L_1(\mathbb{R}^2)$ and $\text{TV}(f) < \infty$.

Whenever $0 < \epsilon < \infty$ we let

$$F_\epsilon(f) = c\text{TV}(f) + F(f) \quad \text{for } f \in \mathcal{F}(\mathbb{R}^2).$$

We call $F_\epsilon$ the total variation regularization of $F$ (with smoothing parameter $\epsilon$). Let

$$m^{loc}_\epsilon(F)$$

be the set of those $f \in \mathcal{F}(\mathbb{R}^2)$ such that $\text{TV}(f) < \infty$ and

$$F_\epsilon(f) \leq F_\epsilon(g)$$

whenever $g \in \mathcal{F}(\mathbb{R}^2)$ and $g(x) = f(x)$ outside some compact subset of $\mathbb{R}^2$.

In the denoising literature $s$ might represent a noisy grayscale image and $f \in m_\epsilon(F)$ would be a “denoising” of $s$.

In this paper we determine $m^{loc}_\epsilon(F)$ for certain $\gamma$ and certain $s$.

In [AW] $m^{loc}_\epsilon(F)$ was determined where, for some $p \in [1, \infty)$, $\gamma(y) = |y|^p / p$ for $0 < y < \infty$ and where $s$ was the indicator function of a square.

### 1.2 The associated binary problems.

The determination of $m_\epsilon(F)$ above reduces to the solution of certain “binary” variational problems as follows.

Let

$$\mathcal{M}(\mathbb{R}^2) = \{D : D \subset \mathbb{R}^2 \text{ and } 1_D \in \mathcal{F}(\mathbb{R}^2)\};$$

thus a subset $D$ of $\mathbb{R}^2$ belongs to $\mathcal{M}(\mathbb{R}^2)$ if and only if $D$ is Lebesgue measurable and $\mathcal{L}^2(D) < \infty$.

Given

$$M : \mathcal{M}(\mathbb{R}^2) \to \mathbb{R} \quad \text{and} \quad 0 < \epsilon < \infty$$

and $0 < \epsilon < \infty$ we let

$$M_\epsilon(E) = c\text{TV}(E) + M(E) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2)$$

and we let

$$n^{loc}_\epsilon(M)$$

be the family of those $D \in \mathcal{M}(\mathbb{R}^2)$ such that $\text{TV}(D) < \infty$ and

$$M_\epsilon(D) \leq M_\epsilon(E)$$

whenever $E \in \mathcal{M}(\mathbb{R}^2)$ and $\mathcal{L}^2((D \cup E) \sim K) = 0$ for some compact subset of $\mathbb{R}^2$; here and in what follows we frequently identify “$E$” with “$1_E$”, the indicator function of $E$. 

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Let $s, \gamma, F, \epsilon$ be as in the preceding subsection. Whenever $0 < y < \infty$ let

$$
\beta(y) = \limsup_{z \to y} \frac{\gamma(z) - \gamma(y)}{z - y}
$$

and let

$$
U_y(E) = \int_E \beta(y - s(x)) \, d\mathcal{L}^2 x \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).
$$

**Definition 1.2.1.** We let

$$
\mathcal{G}(\mathbb{R}^2)
$$

be the family of $\mathcal{L}^2 \times \mathcal{L}^1$ measurable subsets $G$ of $\mathbb{R}^2 \times \mathbb{R}$ such that $(\mathcal{L}^2 \times \mathcal{L}^1)(G) < \infty$ and $\{ y : (x, y) \in G \text{ for some } x \}$ is bounded.

Suppose

$$
0 < \epsilon < \infty.
$$

The following Theorem was proved in [AW].

**Theorem 1.2.1.** We have

$$
f \in m_{\epsilon}(F) \Rightarrow \{ f > y \} \in n_{\epsilon}(U_y) \text{ whenever } 0 < y < \infty.
$$

Moreover, if $G \in \mathcal{G}(\mathbb{R}^2)$,

$$
\{ x : (x, y) \in G \} \in m_{\epsilon}(U_y) \quad \text{for } \mathcal{L}^1 \text{ almost all } y \in (0, \infty)
$$

and

$$
f(x) = \mathcal{L}^1(\{ y : (x, y) \in G \}) \quad \text{for } \mathcal{L}^2 \text{ almost all } x \in \mathbb{R}^2
$$

then $f \in m_{\epsilon}(F)$.

Thus if one can determine $n_{\epsilon}(U_y), 0 < y < \infty$, one has also determined $m_{\epsilon}(F)$.

**Theorem 1.2.2.** Suppose $G \in \mathcal{G}(\mathbb{R}^2)$,

$$
\{ x : (x, y) \in G \} \in m_{\epsilon}(U_y) \quad \text{for } \mathcal{L}^1 \text{ almost all } y \in (0, \infty)
$$

and

$$
u(y) = (U_y)_{\epsilon}(\{ x : (x, y) \in G \}) \quad \text{for } y \in (0, \infty).
$$

Then

$$
\mathcal{L}^1(\{ u < 0 \}) = \sup\{ u < 0 \}.
$$

Moreover, if $F$ is strictly convex,

$$
(\mathcal{L}^2 \times \mathcal{L}^1)(\{(x, y) \in G : u(y) = 0\}) = 0.
$$
Proof. Let \( Y = \mathcal{L}^1(\{u < 0\}) \) and \( Z = \sup\{u < 0\} \). Evidently, \( Y \leq Z \). Let \( H = \{(x, y) \in G : u(y) < 0\} \) and let \( h(x) = \mathcal{L}^1(\{y : (x, y) \in H\}) \) for \( x \in \mathbb{R}^2 \). By the preceding Theorem, \( h \in n_{\text{loc}}^1(F) \). Moreover, \( h(x) \leq Y \) for \( x \in \mathbb{R}^2 \). Suppose \( Y < y < \infty \). Then \( \{h > y\} = \emptyset \) so 
\[
 u(y) = (U_y)_c(\{h > y\}) = (U_y)_c(\emptyset) = 0
\]
so \( Z \leq Y \).

Let \( J = \{(x, y) \in G : y \leq Y\} \) and let \( h(x) = \mathcal{L}^1(\{y : (x, y) \in J\}) \) for \( x \in \mathbb{R}^2 \).

Then \( h \in n_{\text{loc}}^1(F) \) by the preceding Theorem and \( j \leq h \). If \( F \) is strictly convex we must have \( h(x) = j(x) \) for \( L^2 \) almost all \( x \) which implies

\[
0 = \int_{\mathbb{R}^2} h - j \, d\mathcal{L}^2 = (\mathcal{L}^2 \times \mathcal{L}^1)(\{(x, y) \in G : u(y) = 0\}).
\]

\( \square \)

### 1.3 The Rudin-Osher-Fatemi (ROF) functional.

In Section 1.1 let \( \gamma(y) = \frac{1}{2} y^2 \) for \( y \in \mathbb{R} \).

This is the case treated in [ROF]. Whenever \( 0 < y < \infty \) we have

\[
U_y(E) = \int_E y - s(x) \, d\mathcal{L}^2 x \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).
\]  
(1)

In particular, if \( s = 1_S \) for some \( S \in \mathcal{M}(\mathbb{R}^2) \) and \( 0 < y < \infty \)

\[
U_y(E) = y\mathcal{L}^2(E \sim S) - (1 - y)\mathcal{L}^2(E \cap S) \quad \text{whenever } E \in \mathcal{M}(\mathbb{R}^2).
\]  
(2)

It is evident that

\[
n^1_{\text{loc}}(U_y) = \{D \in \mathcal{M}(\mathbb{R}^2) : \mathcal{L}^2(D) = 0\} \quad \text{if } s(x) < y \text{ for } \mathcal{L}^2 \text{ almost all } x.
\]  
(3)

### 1.4 The Chan-Esedoglu (CE) functional.

In Section 1.1 let

\[
\gamma(y) = |y| \quad \text{for } 0 < y < \infty.
\]

This is the case treated in [CE]. For \( 0 < y < \infty \) and \( E \in \mathcal{M}(\mathbb{R}^2) \) we have

\[
U_y(E) = \mathcal{L}^2(E \cap \{y \geq s\}) - \mathcal{L}^2(E \cap \{y < s\}) \\
= \mathcal{L}^2(E \sim \{s \geq y\}) - \mathcal{L}^2(E \cap \{s > y\}) \\
= V_{\{s>y\}}(E)
\]  
(4)

where for any \( S \in \mathcal{M}(\mathbb{R}^2) \) we have set

\[
V_S(D) = \mathcal{L}^2(D \sim S) - \mathcal{L}^2(D \cap S) \quad \text{for } D \in \mathcal{M}(\mathbb{R}^2).
\]

Evidently,

\[
n^1_{\text{loc}}(V_0) = \{D \in \mathcal{M}(\mathbb{R}^2) : \mathcal{L}^2(D) = 0\}.
\]  
(5)
1.5 Some useful definitions and notations.

The first appearance of any term which is about to be defined will always appear in boldface, be displayed or appear inside a Definition.

Let $L^2$ be Lebesgue measure on $\mathbb{R}^2$ and let $H^1$ be one dimensional Hausdorff measure on $\mathbb{R}^2$.

Whenever $a \in \mathbb{R}^2$ and $0 < r < \infty$ we let

$$U(a, r) = \{ x \in \mathbb{R}^2 : |x - a| < r \}, \quad B(a, r) = \{ x \in \mathbb{R}^2 : |x - a| \leq r \}$$

and we let

$$C(a, r) = \{ x \in \mathbb{R}^2 : |x - a| = r \}.$$

We let

$$S^1 = \{ x \in \mathbb{R}^2 : |x| = 1 \}$$

and we let

$$e_1 = (1, 0) \in S^1, \quad e_2 = (0, 1) \in S^1.$$

Whenever $a, b \in \mathbb{R}^2$ we let

$$(a, b) = \{(1-t)a+tr : 0 < t < 1\} \quad \text{and we let} \quad [a, b] = \{(1-t)a+tr : 0 \leq t \leq 1\}.$$

For each $v \in \mathbb{R}^2 \sim \{0\}$ and $t \in \mathbb{R}$ we let

$$h(v, t) = \{ x \in \mathbb{R}^2 : x \bullet v \leq t \} \quad \text{and} \quad l(v, t) = \{ x \in \mathbb{R}^2 : x \bullet v = t \}.$$

For each $(u, z) \in S^1 \times \mathbb{R}$ we let

$$h(u, z) = \{ x \in \mathbb{R}^2 : x \bullet u \leq z \} \quad \text{and we let} \quad H = \{ h(u, z) : (u, z) \in S^1 \times \mathbb{R} \}.$$

For each $\theta \in \mathbb{R}$ we let

$$u(\theta) = (\cos \theta, \sin \theta).$$

Whenever $c \in \mathbb{R}^2$, $0 < r < \infty$, $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$ we let

$$a(c, r, \alpha, \beta) = \{ c + r u(\theta) : \alpha < \theta < \beta \}.$$

Whenever $0 < r < \infty$ we let

$$A(r) = \{ a(c, r, \alpha, \beta) : c \in \mathbb{R}^2, \ 0 < r < \infty, \ \alpha < \beta, \ \alpha, \beta \in \mathbb{R} \ \text{and} \beta - \alpha \leq \pi \};$$

thus $A(r)$ is the family of proper open arcs $A$ of circles of radius $r$ such that the length of $A$ does not exceed $\pi r$. Whenever $A \in A(r)$ we let

$$\text{ends}(A) = (\text{cl} A) \sim A$$

and note that $\text{ends}(A)$ contains exactly two points.

Suppose $A \in A(r)$. Then there is a unique $(c, r, \alpha, \beta)$ such that $0 \leq \alpha < 2\pi$ and $A = a(c, r, \alpha, \beta)$; we let

$$\text{circle}(A) = C(c, r)$$
and we let
\[\text{radius}(A) = r; \quad \text{center}(A) = c; \quad \text{first}(A) = c + ru(\alpha); \quad \text{last}(A) = c + ru(\beta).\]

For each \(i = 1, 2\) we let
\[L_i = \{te_i : t \in \mathbb{R}\}\]
and we let \(\sigma_i\) be reflection across \(L_i\). We let
\[L_\pm = \{x \in \mathbb{R}^2 : x_2 = \pm x_1\}\]
and we let \(\sigma_\pm\) be reflection across \(L_\pm\).

**Definition 1.5.1.** Whenever \(E \in \mathcal{M}(\mathbb{R}^2)\) we let
\[\langle E \rangle\]
be the support of the generalized function corresponding to the indicator function \(1_E\) of \(E\).

### 1.6 Quadratics.

Suppose \(I\) is an open interval,
\[a, b, c : I \to \mathbb{R},\]
\(a\) never vanishes and \(\Delta = b^2 - 4ac > 0\)

Let
\[x_\pm = \frac{-b \pm \sqrt{\Delta}}{2a},\]

then
\[ax_\pm^2 + bx_\pm + c = 0\]
and \(x_- < x_+\). As the reader may easily verify, if \(a, b, c\) are differentiable then so are \(x_\pm\) and
\[\frac{\sqrt{\Delta}}{a} (x_\pm)' = \mp \left( \left( \frac{b}{a} \right)' x_\pm + \left( \frac{c}{a} \right)' \right).\]

### 2 The family \(\Gamma(S, r, s)\), \(0 < r < \infty, 0 < s < \infty\).

**Definition 2.0.1.** Suppose \(E \subset \mathbb{R}^2\). We let
\[\text{reg}(E)\]
be the set of those \(b\) such that \(b \in \text{bdry } E\) and there are open intervals \(I\) and \(J\) containing 0; a continuously differentiable function \(f : I \to J\); and an isometry \(\Phi : \mathbb{R}^2 \to \mathbb{R}^2\) such that \(\Phi(b) = 0, f(0) = 0, f'(0) = 0\) and
\[\Phi[E] \cap (I \times J) = \{(u, v) \in I \times J : v \leq f(u)\}.\]
**Definition 2.0.2.** Suppose $E$ and $S$ are compact subsets of $\mathbb{R}^2$. We let

\[ \text{cmp}(E, S) = \{ A : A \text{ is a nonempty connected component of (bdry} E \sim (\text{bdry} S) \}; \]

\[ \text{int}(E, S) = \{ A \in \text{cmp}(E, S) : A \subset \text{int} S \}; \]

\[ \text{ext}(E, S) = \{ A \in \text{cmp}(E, S) : A \subset \mathbb{R}^2 \sim S \}; \]

**Definition 2.0.3.** Suppose $S$ is a compact subset of $\mathbb{R}^2$, $0 < r \leq \infty$ and $0 < s \leq \infty$. We let

\[ \Gamma(S, r, s) \]

be the family of compact subsets $E$ of $\mathbb{R}^2$ such that the following conditions hold:

1. (Γ0) $\text{bdry} E = \text{reg}(E)$;
2. (Γ1) $E$ is a subset of the convex hull of $S$;
3. (Γ2) If $A \in \text{int}(E, S)$ then $A \in A(r)$ and the length of $A$ does not exceed $\pi r$ if $r < \infty$ and $A$ is a subset of a line if $r = \infty$;
4. (Γ3) If $A \in \text{int}(E, S)$, $r < \infty$, and $c$ is such that $A \subset C(c, r)$ then there is an open subset $U$ of $\text{int} S$ such that $A \subset U$ and $U \cap E = U \cap B(c, r)$. 
5. (Γ4) If $A \in \text{ext}(E, S)$ then $A \in A(s)$ and the length of $A$ does not exceed $\pi s$ if $s < \infty$ and $A$ is a subset of straight line if $s = \infty$.
6. (Γ5) If $A \in \text{ext}(E, S)$, $s < \infty$ and $c$ is such that $A \subset C(c, r)$ then there is an open subset $U$ of $\mathbb{R}^2 \sim S$ such that $A \subset U$ and $U \cap E = U \sim U(c, s)$.

Note that $\emptyset \in \Gamma(S, r, s)$.

### 2.1 Some results from [AW].

**Corollary 2.1.1.** Suppose

1. $\gamma, s$ and $F$ are as in Section 1.1 and $s$ is the indicator function of a compact subset $S$ of $\mathbb{R}^2$;
2. $U_y$, $0 < y < \infty$ and $\beta$ are as in Section 1.2;
3. $\gamma$ is smooth away from 0 and $\gamma'(y) \neq 0$ if $y \in \mathbb{R} \sim \{0\}$;
4. $0 < y < \infty$ and $E \in \mathbf{n}_x^{loc}(U_y)$.

Then

\[ L^2((< E >, \sim E) \cup (E \sim < E >)) = 0 \]

and

\[ < E > \begin{cases} \in \mathbf{n}_x(U_y) \cap \Gamma \left( S, \frac{e}{\gamma(1-y)}, \frac{\gamma'}{\gamma(y)} \right) & \text{if } 0 < y < 1; \\
\in \mathbf{n}_x(U_y) \cap \Gamma \left( S, \lim_{y \to 1} \frac{e}{\gamma(1-y)}, \frac{\gamma'}{\gamma(y)} \right) & \text{if } y = 1; \\
= \emptyset & \text{if } y > 1. \end{cases} \]
Definition 2.1.1. We say a subset $E$ of $\mathbb{R}^n$ is special if
\[ \sigma[\text{Tan}(\text{bdry} E, e)] = \text{Tan}(\text{bdry} E, e) \]
whenever $\sigma$ is a rigid motion of $\mathbb{R}^2$, $e$ is an accumulation point of $E$ and $\sigma(e) = e$.

Theorem 2.1.1. Suppose $S \in \mathcal{M}(\mathbb{R}^2)$, $0 < \epsilon < \infty$ and $E \in n^\text{loc}_\epsilon(V_S)$. Then $< E >$ is special.

Proof. Suppose $e$ is an accumulation point of $< E >$ and $\sigma$ is a rigid motion of $\mathbb{R}^2$ and $\sigma(e) = e$. Since $n^\text{loc}_\epsilon(V_S) = n^\text{loc}_\epsilon(N_S)$ where $N_S$ is as in [AW, ??] we have $F = \sigma[< E >] \cap < E > \in n^\text{loc}_\epsilon(V_S)$.

Suppose, contrary to the Theorem, $\text{Tan}(\text{bdry} < E >, e) \neq \sigma[\text{Tan}(\text{bdry} < E >, e)]$. Then $\text{Tan}(< E >, e) \cap \sigma[\text{Tan}(< E >, e)]$ is not a closed halfspace in $\mathbb{R}^2$, $< F > = F$, $e \in \text{bdry} < F >$ and $\text{Tan}(< F >, e)$ is a closed halfspace in $\mathbb{R}^2$ by ???. But
\[ \text{Tan}(< F >, e) = \text{Tan}(< E >, e) \cap \sigma[\text{Tan}(< E >, e)] = \text{Tan}(< E > \cap \sigma[\text{Tan}(< E >, e)]). \]

\[ \square \]

2.2 Some results on $\Gamma(S, r, s)$.

Throughout this section we assume that $S$ is a compact subset of $\mathbb{R}^2$; $0 < r < \infty$; $0 < s < \infty$; and that $E \in \Gamma(S, r, s)$.

Proposition 2.2.1. Suppose $A \in \text{cmp}(E, S)$; $b \in \text{ends}(A)$; $u$ and $v$ are such that
\[ \text{Nor}(E, b) \cap S^1 = \{u\}; \quad \text{Tan}(A, b) \cap S^1 = \{v\}; \]
and
\[ H = \{x \in \mathbb{R}^2 : (x - b) \cdot v \geq 0\}. \]

If $A \in \text{int}(E, S)$ then
\[ A \subset C(b - ru, r) \]
and there is $\delta \in (0, r)$ such that
\[ E \cap H \cap U(b, \delta) = B(b - ru, r) \cap H \cap U(b, \delta). \]

If $A \in \text{ext}(E, S)$ then
\[ A \subset C(b + su, s) \]
and there is $\delta \in (0, s)$ such that
\[ E \cap H \cap U(b, \delta) = (H \sim U(b + su, s)) \cap U(b, \delta). \]
Proof. Since \( b \in \text{reg}(E) \) and \( A \subset \text{bdry } E \) we find that \( \text{Tan}(A, b) \subset \text{Tan}(\text{bdry } E, b) \) and \( u \cdot v = 0 \). Thus there are \( \eta, J, f, g, \Phi \) such that \( 0 < \eta < \infty \); \( J \) is an open interval in \( \mathbb{R} \) containing \( 0 \); \( f : (-\eta, \eta) \to J \); \( g : (0, \eta) \to J \); \( f \) and \( g \) are continuously differentiable; \( f(0) = 0 = \lim_{t \downarrow 0} g(t) \); \( f'(0) = 0 = \lim_{t \downarrow 0} g'(t) \); \( \Phi \) is a an isometry of \( \mathbb{R}^2 \);

\[
\Phi[E] \cap ((-\eta, \eta) \times J) = \{(t, u) \in (-\eta, \eta) \times J : u \leq f(t)\}
\]

and

\[
\Phi[A] \cap ((0, \eta) \times J) = g.
\]

Since \( A \subset \text{bdry } E \) we find that \( g \subset f \). We leave it to the reader to use (Γ3) and (Γ5) to supply the remaining details of the proof. \( \Box \)

Proposition 2.2.2. Suppose \( b \in \text{bdry } S \) and

\[
\mathcal{A} = \{A \in \text{cmp}(E, S) : b \in \text{ends}(A)\}.
\]

Then

(i) \( \mathcal{A} \) has at most two members;

(ii) if \( b \in \text{bdry } E \) and either \( b \) is an isolated point of \( \text{bdry } S \) or \( b \) is an accumulation point of \( \text{bdry } S \) and

\[
\text{Tan}(\text{bdry } E, b) \cap \text{Tan}(\text{bdry } S, b) = \{0\}
\]

then \( \mathcal{A} \) has exactly two members;

(iii) if \( b \in (\text{bdry } E) \cap (\text{reg}(S)) \) and

\[
\text{Tan}(\text{bdry } E, b) \cap \text{Tan}(\text{bdry } S, b) = \{0\}
\]

then there are \( A \in \text{int}(E, S) \) and \( B \in \text{ext}(E, S) \) such that \( \mathcal{A} = \{A, B\} \).

Proof. The previous Proposition directly implies (i).

Suppose the hypotheses of (ii) hold. Let \( I, J, f, \Phi \) be as in (2.0.1). Shrinking \( I \) and \( J \) if necessary we may assume that \( \Phi[\text{bdry } S] \cap f = \{(0, 0)\} \). Let \( I^+ = \{t \in I : t > 0\} \) and let \( I^- = \{t \in I : t < 0\} \). Then \( \Phi^{-1}[I^+] \) and \( \Phi^{-1}[I^-] \) are connected subsets of \( \mathbb{R}^2 \sim \text{bdry } S \) so there are \( A^+ \) and \( A^- \) in \( \text{cmp}(E, S) \) such that \( \Phi^{-1}[I^+] \subset A^+ \) and \( \Phi^{-1}[I^-] \subset A^- \). Given the length restriction on each of \( A^+ \) and \( A^- \) we find that \( A^+ \neq A^- \). Thus (ii) holds.

We leave it as a simple exercise for the reader to prove (iii). \( \Box \)

Proposition 2.2.3. Suppose \( S \) is convex and \( a \in E \cap (\text{bdry } S) \). Then \( a \in \text{bdry } E \) and

\[
E \subset S \subset a + \text{Tan}(S, a).
\]

Moreover, if \( A \in \text{int}(E, S) \), \( a \in \text{ends}(A) \) and \( c \) is the center of the circle containing \( A \) then

\[
S \subset \{x \in \mathbb{R}^2 : (x - a) \cdot (c - a) \geq 0\}.
\]
Proof. $E \subset S$ by (T1) and this implies $Tan(E, a) \subset Tan(S, a)$. Since $a \in bdry S$ and $S$ is convex $Tan(S, b)$ is contained in a closed halfspace. Thus $Tan(S, b) = Tan(E, b)$. This in turn implies $b \in bdry E$. The finaly assertion of the Proposition directly follows.

Lemma 2.2.1. Suppose $S$ is convex and, for $i = 1, 2$, $A_i \in int(E, S)$, $a_i \in ends(A_i)$ and $c_i$ is the center of the circle containing $A_i$. Then

$$(a_i - c_i) \cdot (c_j - c_i) \leq 0 \text{ whenever } \{i, j\} = \{1, 2\}.$$

Proof. Suppose $\{i, j\} = \{1, 2\}$. From (??) and the fact that $|a_i - c_i| = r = |a_j - c_j|$ we obtain

$$0 \geq (a_j - a_i) \cdot (a_i - c_i)$$

$$= (c_j + (a_j - c_j) - (c_i + (a_i - c_i))) \cdot (a_i - c_i)$$

$$= (c_j - c_i) \cdot (a_i - c_i) + (a_j - c_j) \cdot (a_i - c_i) - |a_i - c_i|^2$$

$$\geq (c_j - c_i) \cdot (a_i - c_i).$$

Proposition 2.2.4. Suppose $S$ is convex; $a, b, c, A$ are such that $A \in int(E, S)$, $A \subset C(c, r)$, $ends(A) = \{a, b\}$; and $V = \{t(x - c) : x \in cl A \text{ and } t > 1\}$. Then exactly one of the following holds:

(i) $E \cap V = \emptyset$.

(ii) The length of $A$ equals $\pi r$ and there are $a', b', c', A'$ such that $A' \in int(E, S)$, $A' \subset C(c', r)$, $ends(A') = \{a', b'\}$ and such that, for some $q > 2$,

$$a' = a + q(d - c), \quad b' = b + q(d - c), \quad c' = c + q(d - c).$$

where $d$ is the midpoint of $A$.

Furthermore, if (ii) holds we have

$$E \cap G = \emptyset$$

where $G$ is the union of the segments $(e, e')$ such that $e \in A$, $e' \in A'$ and $(e, e')$ is parallel to the line containing the centers of $A$ and $A'$.

Proof. Since $S$ is compact and convex, whenever $L$ is a line and $y \in L \cap int S$ there are unique $x, z, t$ such that $\{x, z\} \subset L \cap bdry S$, $0 < t < 1$ and $y = (1 - t)x + tz$. Since $A \subset int S$ it follows that there is one and only one function $v : A \to int V \cap bdry S$ such that

$$bdry S \cap \{c + t(x - c) : 1 < t < \infty\} = \{v(x)\} \text{ for } x \in A.$$

Owing to the convexity of $S$ we find that

(iv) $c + ru \in A$ whenever $x \in A$ and and $u \in S^1 \cap Nor(S, v(x))$.  

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Suppose \( E \cap V \neq \emptyset \). Owing to the compactness and regularity properties of \( E \) there exists \( d' \in V \cap \text{bdry} \ E \) such that
\[
|d' - c| = \min \{|x - c| : x \in E \cap V\} > r.
\]
It follows from \((\Gamma 3)\) that
\[
\text{Nor}(E, d') = \{t(c - d') : 0 \leq t < \infty\}.
\]
From (iii) we have
\[
d' \in E \subset \{x \in \mathbb{R}^2 : (x - a) \bullet (a - c) \leq 0 \text{ and } (x - b) \bullet (b - c) \leq 0\}
\]
which implies \( d' \in \text{int} \ V \) so there is \( d \in A \) such that \( d' = c + t(d - c) \) for some \( t \in (1, \infty) \).

Were it the case that \( d' \in \text{bdry} \ S \) we could infer from the preceding Proposition that \( \text{Nor}(S, d') = \text{Nor}(E, d') \) which is incompatible with (iv). Thus \( d' \notin \text{bdry} \ S \). Since \( d' \in E \subset S \) we infer that \( d' \in \text{int} \ S \). Thus there is \( A' \in \text{int}(E, S) \) such that \( d' \in A' \). It follows from \((\Gamma 3)\) that \( A' \subset C(c', r) \) where \( c' = c + u(d - c) \) for some \( u \in (t + r, \infty) \). Thus \( |c' - c| > 2r \).

Let \( a', b' \) such that \( \text{ends}(A') = \{a', b'\} \). From the preceding Lemma we have
\[
(a' - c') \bullet (c' - c) \leq 0 \quad \text{and} \quad (b' - c') \bullet (c' - c) \leq 0.
\]
Since the length of \( A' \) does not exceed \( \pi r \) we infer that \( A' \) is a semicircle with midpoint \( d' \). By a similar argument making use we find that \( A \) is a semicircle with midpoint \( d \). (ii) now follows.

Let \( G \) be as in the final conclusion of the Proposition. From the convexity of \( S \) we infer that the interior of the rectangle containing the points \( a, b, a', b' \) is a subset of \( \text{int} \ G \) so \( G \subset \text{int} \ G \). Suppose, contrary to the conclusion of the Proposition, \( p \in E \cap G \). Since \( p \notin \text{bdry} \ S \) there is \( B \in \text{int}(E, S) \) such that \( p \in E \). Since \( B \) cannot meet \( A \cup A' \) we must have \( \text{ends}(B) \subset [a, a'] \cup [b, b'] \). Since \( (d, d') \cap \text{bdry} \ E = \emptyset \) we infer that either \( \text{ends}(B) \subset [a, b] \) or \( \text{ends}(B) \subset [a', b'] \). This is impossible given \((\Gamma 1)-(\Gamma 5)\). \( \square \)

**Proposition 2.2.5.** Suppose \( S \) is a square with center \( c \) and half side length \( l \). Then
\[
\Gamma(S, r, s) = \begin{cases} \emptyset & \text{if } r > l, \\ \emptyset, S \cap \mathbb{B}(c, \sqrt{(l-s)^2+l^2}) & \text{if } r \leq l. \end{cases}
\]

**Proof.** Let \( V \) be the set of vertices of \( S \) and suppose \( E \in \Gamma(S, r, s) \sim \{\emptyset\} \). For each vertex \( v \in V \) let \( t_v = \sup\{(x - c) \bullet (v - c) : x \in E\} \). Since \( E \) is compact we may choose for each vertex \( v \in V \) a point \( e_v \in E \) such that \( t_v = (e_v - c) \bullet (v - c) \).

Suppose \( v \in V \). By \((\Gamma 0)\) we have \( e_v \in \text{bdry} \ S \cap \text{int} \ S \) so there is \( A_v \in \text{int}(E, S) \) such that \( e_v \in A_v \). By \((\Gamma 3)\), \( |e_v - c| \mathbf{n}_E(e_v) = |e_v - c| \) and \( A \subset \{x \in \mathbb{R}^2 : (x - e_v) \bullet (c - e_v) \geq 0\} \). Let \( C_v \) be the circle containing \( A_v \). By \((\Gamma 0)\) and \((\Gamma 1)\), \( C_v \) meets the interior of an edge of \( \text{bdry} \ S \) tangentially at each point of \( \text{ends}(A_v) \). Since the length of \( A \) cannot exceed \( \pi r \) the Proposition follows. \( \square \)
2.2.1 A basic theorem.

**Theorem 2.2.1.** Suppose

(i) $S$ is a compact subset of $\mathbb{R}^2$;

(ii) $Y$ is a line or a circle;

(iii) $G$ is the set of $a \in \mathbb{R}^2$ such that either $a \notin \text{bdry } S$ or $a \in \text{bdry } S$ and there is an open subset $W$ of $\mathbb{R}^2$ such that $a \in W$ and $W \cap \text{bdry } S = W \cap Y$;

(iv) $0 < r < \infty, 0 < s < \infty$ and $E \in \Gamma(S, r, s)$;

(v) $X$ is a connected component of $G \cap \text{bdry } E$;

\[ X \sim \text{bdry } S \neq \emptyset \quad \text{and} \quad X \cap \text{bdry } S \neq \emptyset; \]

(vi)

\[
P = \{a \in X \cap Y : \text{Tan}(X, a) = \text{Tan}(Y, a)\}; \\
Q = \{a \in X \cap Y : \text{Tan}(X, a) \cap \text{Tan}(Y, a) = \{0\}\}; \\
\mathcal{R} = \{A \in \text{cmp}(E, S) : A \subset X\}; \\
\mathcal{R}_1 = \{A \in \mathcal{R} : \text{card}(Q \cap \text{ends}(A)) = 1 \text{ and card } (\text{ends}(A) \sim X) = 1\}; \\
\mathcal{R}_2 = \{A \in \mathcal{R} : \text{card}(Q \cap \text{ends}(A)) = 2\}; \\
\mathcal{R}_3 = \{A \in \mathcal{R} : \text{card}(P \cap \text{ends}(A)) = 1 \text{ and card } (\text{ends}(A) \sim X) = 1\}; \\
\mathcal{R}_i = \cup \mathcal{R}_i, \ i = 1, 2, 3.
\]

Then

(vii) $G$ is open, $G \cap \text{bdry } S = G \cap Y$ and $X$ is a relatively open subset of $\text{bdry } E$;

(viii) $X$ is homeomorphic to a line or a circle;

(ix) $\mathcal{R} = \{A \in \text{cmp}(E, S) : X \cap \text{cl } A \neq \emptyset\}$

(x) $X \sim Y = \cup \mathcal{R}$;

(xi) $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$;

(xii) $a \in Q$ if and only if there are $A \in (\mathcal{R}_1 \cup \mathcal{R}_2) \cap \text{int}(E, S)$ and $B \in (\mathcal{R}_1 \cup \mathcal{R}_2) \cap \text{ext}(E, S)$ such that

\[ \text{ends}(A) \cap \text{ends}(B) = \{a\}; \]

(xiii) if $A, B \in \mathcal{R}_1$, $A \neq B$ and either $A, B \in \text{int}(E, S)$ or $A, B \in \text{ext}(E, S)$ then there are $a \in \text{ends}(A)$ and $b \in \text{ends}(B)$ such that $a \neq b$ and $\rho(C) = D$ where $\rho$ is reflection across the perpendicular bisector of the segment joining $a$ to $b$ and $C$ and $D$ are the circles containing $A$ and $B$, respectively;
(xiv) If \( A, B \in \mathcal{R}_2 \), \( A \neq B \) and either \( A, B \in \text{int}(E, S) \) or \( A, B \in \text{ext}(E, S) \) then there are \( a \in \text{ends}(A) \) and \( b \in \text{ends}(B) \) such that \( a \neq b \) and \( \rho[A] = B \) where \( \rho \) is reflection across the perpendicular bisector of the segment joining \( a \) to \( b \).

(xv) \( P \) is connected.

Moreover, exactly one of the following holds:

(I) \( \mathcal{R}_1 = \emptyset, \mathcal{R}_2 = \emptyset, 1 \leq \text{card} \mathcal{R}_3 \leq 2, P = X \cap Y, Q = \emptyset, X = P \cup \mathcal{R}_3 \) and \( X \) is homeomorphic to a line;

(II) \( \text{card} \mathcal{R}_1 = 2, \mathcal{R}_3 = \emptyset, P = \emptyset, Q = X \cap Y, X = R_1 \cup R_2 \cup Q \) and \( X \) is homeomorphic to a line;

(III) \( \mathcal{R}_1 = \emptyset, \text{card} \mathcal{R}_2 \geq 4, \mathcal{R}_3 = \emptyset, P = \emptyset, Q = X \cap Y, X = R_2 \cup Q, Y \) is a circle and \( X \) is homeomorphic to a circle.

**Proof.** (vii) holds. It is obvious that \( G \) is open and that \( G \cap \text{bdry} S = G \cap Y \). Since \( G \) is open we infer that \( G \cap \text{bdry} E \) is open relative to \( \text{bdry} E \). Since \( \text{bdry} E \) is locally connected we infer that \( X \) is a relatively open subset \( \text{bdry} E \).

(viii) holds. Any relatively open connected subset of \( \text{bdry} E \) is homeomorphic to a line or a circle owing to (I1).

(ix) holds. Suppose \( A \in \text{cmp}(E, S) \) and \( X \cap \text{cl} A \neq \emptyset \). If \( X \cap A \neq \emptyset \) then \( X \cup A \) is a connected subset of \( G \cap \text{bdry} E \) so \( X \cup A \subset X \) so \( A \subset X \). If \( X \cap A = \emptyset \) then there is \( a \in X \cap \text{ends}(A) \) so \( X \cup (A \cup \{a\}) \) is a connected subset of \( G \cap \text{bdry} E \) so \( X \cup (A \cup \{a\}) \subset X \) so \( A \subset X \).

(x) holds. Suppose \( A \in \mathcal{R} \). Since \( A \subset X \), \( A \cap \text{bdry} S = \emptyset \) and \( G \cap \text{bdry} S = G \cap Y \) we infer that \( A \subset X \sim Y \).

Suppose \( a \in X \sim Y \). Then \( a \notin \text{bdry} S \) so there is \( A \in \text{cmp}(E, S) \) such that \( a \in A \). (ix) then implies \( A \in \mathcal{R} \).

(xi) holds. Suppose \( A \in \mathcal{R} \) and let \( C \) be the circle containing \( A \). Were it the case that \( \text{ends}(A) \cap X = \emptyset \) we would have \( A = X \) since \( X \) is homeomorphic to a line or a circle and this would imply \( X \cap \text{bdry} S = \emptyset \) which is incompatible with (v). Thus there exist \( a, b \) such that \( \text{ends}(A) = \{a, b\} \) and \( a \in X \times Y \). Note that if \( b \notin Y \) then \( b \notin G \) since \( G \cap \text{bdry} S = G \cap Y \) and \( b \in \text{bdry} S \).

Since \( X \) is a relatively open subset of \( \text{bdry} E \) we find that \( \text{Tan}(X, a) \) is a line. Thus either \( a \in P \) or \( a \in Q \).

In case \( a \in P \) we have \( C \cap Y = \{a\} \) so \( b \notin Y \). But then \( b \notin G \) so \( A \in \mathcal{R}_3 \).

In case \( a \in Q \) we have that \( C \) intersects \( Y \) transversely in two points. If \( b \notin Y \) then \( b \notin G \) so \( A \in \mathcal{R}_1 \). If \( b \in Y \) then \( b \in Q \) and \( A \in \mathcal{R}_2 \).

(xii) holds. This is a direct consequence of Proposition 2.2.2(iii).

**Definition 2.2.1.** We say a subfamily \( C \) of \( \mathcal{R} \) is a chain if \( C \) is finite and finite and \( \cup \{\text{cl} A : A \in C\} \) is connected.

**Lemma 2.2.2.** Suppose \( C \) is a chain. Then there exist a positive integer \( N \), a univalent function

\[
A : \{1, \ldots, N\} \to \mathcal{R}
\]

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with range $C$ and a function
\[ a : \{0, 1, \ldots, N\} \to X \]
such that
\[ \text{ends}(A_i) = \{a_{i-1}, a_i\} \quad \text{whenever} \quad i = 1, \ldots, N. \]

**Proof.** This is a straightforward consequence of the regularity of $E$. \qed

**Lemma 2.2.3.** Suppose

(i) $C$ is a chain and $C$ has exactly three members;

(ii) $A, a$ are as in Lemma 2.2.2;

(iii) for each $i = 1, 2, 3$, $C_i$ is the circle containing $A_i$ and $B_i$ is the connected component of $C_i \sim Y$ such that $A_i \subset B_i$;

(iv) $\rho_2$ is reflection across the perpendicular bisector of $[a_1, a_2]$.

Then $\rho_2[B_1] = B_3$ and $\rho_2[C_1] = C_3$.

**Proof.** Since $X$ is homeomorphic to a line or a circle we find that $\{a_1, a_2\} \subset X$. This implies that $A_i \in R_1 \cup R_2$, $i = 1, 2, 3$. Thus the circles $C_2$ and $Y$ meet transversely at $a_1$ and $a_2$ and this implies $\rho_2[\text{Tan}(A_2, a_1)] = \text{Tan}(A_2, a_2)$. Since $\text{Tan}(A_1, a_1) = -\text{Tan}(A_2, a_1)$ and $\text{Tan}(A_3, a_2) = -\text{Tan}(A_2, a_1)$ we find that $\rho_2[\text{Tan}(A_0, a_1)] = \text{Tan}(A_3, a_2)$. This in turn implies the Lemma. \qed

In particular, (xiii) and (xiv) hold.

**Corollary 2.2.1.** The union of two chains is a chain.

**Proof.** This is an easy consequence of the fact that $X$ is homeomorphic to a line or a circle. \qed

**Corollary 2.2.2.** Any chain is a subfamily of a unique maximal chain.

**Proof.** It follows from the preceding Lemma that if $C$ is a chain there are positive real numbers $l_i$ and $l_e$ such that if $A \in C \cap R_2$ then the length of $A$ equals $l_i$ if $A \in \text{int}(E, S)$ and equals $l_e$ if $A \in \text{ext}(E, S)$. It follows as well from the preceding Lemma that
\[ |\text{card}C \cap \text{int}(E, S) - \text{card}C \cap \text{ext}(E, S)| \leq 1. \]

We conclude that $(\text{card}C - 2) \min\{l_i, l_e\}$ cannot exceed the length of $X$. Thus there is at least one maximal chain. \qed

**Lemma 2.2.4.** Suppose $Y$ is a circle, $c$ is the center of $Y$, $A \in R_2$, $A$ does not meet the disc whose boundary is $Y$, $H$ is the closed halfspace such that $\text{ends}(A) \subset \text{bdry} H$ and $A \subset H$. Then $c \notin H$. 

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Proof. If \( c \in H \) the the length of \( A \) would exceed \( \pi \) times the radius of the circle containing \( A \).

Lemma 2.2.5. Suppose \( \mathcal{R}_3 \neq \emptyset \). Then (I) holds.

The closure of each member of \( \mathcal{R}_3 \) meets the complement of \( X \). Since \( X \) is homeomorphic to a line or a circle, it follows that \( X \) is homeomorphic to a line, \( \mathcal{R}_3 \) has at most two members and \( P = X \sim \cup \mathcal{R}_3 \) is connected. Thus (I) holds.

Lemma 2.2.6. Suppose \( \mathcal{R}_3 = \emptyset \). Then (II) or (III) holds.

Proof. Let \( C \) be a maximal chain and let \( a, A, N \) be as in Lemma 2.2.2.

Suppose \( X \cap \{a_0, a_N\} = \emptyset \). It follows that \( X \) is homeomorphic to a line and that \( X \sim (A_0 \cup A_N) \) is connected. Thus (II) holds.

Suppose \( \{0, N\} = \{i, j\} \) and \( a_i \in X \). Then \( a_i \in X \cap \text{bdry}E \subset Y \). We cannot have \( \text{Tan}(X, a_i) = \text{Tan}(Y, a_i) \) since \( \mathcal{R}_3 = \emptyset \). Thus \( a_i \in Q \). By Proposition 2.2.2 there is \( B \in \text{cmp}(E, S) \) such that \( a_i \in \text{ends}(B) \). Since \( C \) is maximal and \( X \) is homeomorphic to a line or a circle we have \( B = A_j \) and \( a_i = a_j \). Using the preceding Lemma to infer that \( N \geq 4 \) we see that (III) holds.

3 Minimizers for the Chan-Esedoglu regularization under certain convexity assumptions.

**Definition 3.0.2.** Whenever \( 0 < \epsilon < \infty \) and \( S \subset \mathbb{R}^2 \) we let

\[
b_\epsilon(S) = \bigcup \{ B(a, \epsilon) : a \in \mathbb{R}^2 \text{ and } B(a, \epsilon) \subset S \}.
\]

Our main objective in this section is to prove the following Theorem.

**Theorem 3.0.2.** Suppose \( 0 < \epsilon < \infty \) and \( S \) is a compact convex subset of \( \mathbb{R}^2 \). Then one of the following holds:

(i) \( n^{\text{loc}}_\epsilon(V_S) = \{\emptyset\} \);

(ii) \( b_\epsilon(S) \neq \emptyset \) and \( n^{\text{loc}}_\epsilon(V_S) = \{\emptyset, b_\epsilon(S)\} \);

(iii) \( b_\epsilon(S) \neq \emptyset \) and \( n^{\text{loc}}_\epsilon(V_S) = \{b_\epsilon(S)\} \).

As simple examples which are given below show, all three of these alternatives occur.

**Remark 3.0.1.** Suppose, for each \( i = 1, 2 \), \( S_i \) is a compact convex subset of \( \mathbb{R}^2 \) and \( S_1 \subset S_2 \). Suppose \( 0 < \epsilon < \infty \). Evidently, \( b_\epsilon(S_1) \subset b_\epsilon(S_2) \).

Suppose (ii) of the preceding Theorem holds with \( S = S_1 \). Then

\[
(V_{S_1})_\epsilon(b_\epsilon(S_1)) = (V_{S_1}(\emptyset)) = 0
\]

\[S_1 \subset S_2 \Rightarrow D_1 \subset D_2.\]
Proof. Suppose \( S_1 \subset S_2 \) and \( [D_1] \neq 0 \). Then, as \( D_1 \subset S_1 \subset S_2 \) we have
\[
V_{S_2}(D_1) = |D_1 \sim S_2| - |D_1 \cap S_2| = |D_1 \sim S_1| - |D_1 \cap S_1| = V_{S_1}(D_1)
\]
so
\[
cTV(D_1) + V_{S_2}(D_1) = cTV(D_1) + V_{S_1}(D_1) \leq cTV(\emptyset) + V_{S_1}(\emptyset) = 0 \leq cTV(\emptyset) + V_{S_2}(\emptyset).
\]
Were it the case that \( [D_2] = 0 \) we would have
\[
cTV(D_1) + V_{S_2}(D_1) = 0 = cTV(D_2) + V_{S_2}(D_2)
\]
so \( D_1 \in n_{loc}^\epsilon(V_{S_2}) \) so \( D_1 = (S_2) \) by ??.

Keeping in mind ?? we obtain the following.

**Corollary 3.0.3.** Suppose \( s \in F(\mathbb{R}^2), s \geq 0, \{s \geq y\} \) is convex whenever \( y > 0 \) and \( f \in \mathfrak{m}(F) \) where \( F \) is as in the introduction. Then for any \( y > 0 \) we have that \( \{f \geq y\} \) is either essentially empty or essentially equal to \( \{s \geq y\} \).

**Example 3.0.1.** Suppose \( S \) is a compact convex subset of \( \mathbb{R}^n \), \( 0 \in \text{int}S \) and \( \text{bdry}S \) is polygonal. Let \( V \) be the set of vertices of \( \text{bdry}S \) and let \( E \) be the family of open edges of \( \text{bdry}S \). For each \( t \in (0, \infty) \) let \( S_t = \{tx : x \in S\} \).

Suppose \( 0 < \epsilon < \infty \) and \( t \in (\epsilon, \infty) \). Then
\[
b_\epsilon(S_t) = \{x \in \mathbb{R}^2 : \text{dist} \ (x, S_{t-\epsilon}) \leq \epsilon\}.
\]
Moreover, \( b_\epsilon(S_t) \) is the union of \( S_{t-\epsilon} \); the rectangles
\[
\{(t-\epsilon)e + v : e \in E \text{ and } v \in B(0, \epsilon) \cap \text{Nor}(S, e)\}
\]
corresponding to \( E \in E \); and the disk sectors
\[
(t-\epsilon)v + \left(B(0, \epsilon) \cap \text{Nor}(S, v)\right)
\]
corresponding to \( v \in V \). Thus
\[
TV(b_\epsilon(S_t)) = \sum_{E \in E} (t-\epsilon)H^1(E) + \sum_{v \in V} cH^1(S^1 \cap \text{Nor}(S, v))
\]
and
\[
L^2(b_\epsilon(S_t)) = (t-\epsilon)^2L^2(S) + \sum_{E \in E} (t-\epsilon)cH^1(E) + \sum_{v \in V} c^2L^2(\{0, 1\} \cap \text{Nor}(S, v))
\]
Since
\[
\sum_{v \in V} H^1(S^1 \cap \text{Nor}(S, v)) = 2\pi \quad \text{and} \quad \sum_{v \in V} L^2(\{0, 1\} \cap \text{Nor}(S, v)) = \pi
\]
We find that
\[ \epsilon \text{TV}(b_{\epsilon}(S_t)) - \mathcal{L}^2(b_{\epsilon}(S_t)) = \pi \epsilon^2 - (t - \epsilon)^2 \mathcal{L}^2(S). \]

Let
\[ T_{\epsilon} = \epsilon \left( 1 + \sqrt{\frac{\pi}{\mathcal{L}^2(S)}} \right). \]

Thus if \( D \) is a Lebesgue measurable subset of \( \mathbb{R}^2 \) we find that
\[ D \in n_{\epsilon}^{loc}(S_t) \Leftrightarrow \Sigma(D, \emptyset) = 0 \text{ if } t < T_{\epsilon}; \]
\[ D \in n_{\epsilon}^{loc}(S_t) \Leftrightarrow \Sigma(D, \emptyset) = 0 \text{ or } \Sigma(D, C_t) = 0 \text{ if } t = T_{\epsilon}; \]
\[ D \in n_{\epsilon}^{loc}(S_t) \Leftrightarrow \Sigma(D, C_t) = 0 \text{ if } T_{\epsilon} < t. \]

Note that as \( T_{\epsilon} \to 2\epsilon \) as \( \mathcal{L}^2(S) \) approaches \( \pi \). This is consistent with the observation in [CE] that \( D \in n_{1}^{loc}(B(0, \epsilon)) \) if and only if \( \Sigma(E, \emptyset) = 0 \) or \( \Sigma(E, B(0, 2\epsilon)) = 0 \).

### 3.1 Some facts about \( B_1(\mathbb{R}^2) \).

**Definition 3.1.1.** We let
\[ w(m) = \sqrt{1 + m^2} \text{ for } m \in \mathbb{R}. \]

Note that
\[ w'(m) = \frac{m}{w(m)}, \quad \text{that} \quad w''(m) = \frac{1}{w(m)^3}, \quad m \in \mathbb{R}, \]
and that if \( I \) is an open interval in \( \mathbb{R} \) and \( f : I \to \mathbb{R} \) is twice differentiable then \((w \circ f')(t)\) is the curvature of \( f \) at \((t, f(t))\) whenever \( t \in I \).

**Theorem 3.1.1.** Suppose
(i) \( E \in B_1(\mathbb{R}^2) \) and \( E \) is compact;
(ii) \( a \in \text{bdry } E, u, v \in S^1, u \in \text{Tan(} \text{bdry } E, a \text{)} \) and \(-v \in \text{Nor}(E, a)\);
(iii) \( C \) is the connected component of \( a \) in \( \{x \in \text{bdry } E : |(x - a) \cdot u| < 1\} \).

Then there is a continuously differentiable function
\[ f : (-1, 1) \to \mathbb{R} \]
such that
(iv) \( |f(t)| \leq 1 - \sqrt{1-t^2} \) and \( |f'(t)| \leq \frac{|t|}{\sqrt{1-t}} \) whenever \(-1 < t < 1; \)
(v) \( \text{Lip (} w' \circ f' \text{)} \leq 1; \)
(vi) \( C = \{ a + t(x - a) + f(t)(v - a) : -1 < t < 1 \} \).

Proof. We may assume without loss of generality that \( a = 0, u = e_1 \) and \( v = e_2 \).

Let \( \mathcal{G} \) be the family of ordered pairs \((J, g)\) such that \( J \) is a subinterval of \((-1, 1)\) containing \( 0 \), \( g : J \to \mathbb{R} \) is continuously differentiable, and \( g \subset C \). Owing to the regularity of \( \text{bdry} \ E \) as established in [AW, ?] we find that if \((J_i, g_i) \in \mathcal{G}, i = 1, 2, \) then \((J_1 \cup J_2, g_1 \cup g_2) \in \mathcal{G}\) from which it follows that there is \((I, f) \in \mathcal{G}\) such that \( g \subset f \) whenever \((J, g) \in \mathcal{G}\).

By the Remark following the statement of [AW, ?] we find that \( \text{Lip}(w' \circ f') \leq 1 \). This implies that

\[
|w'(f'(t))| = |w'(f'(t)) - w'(f'(0))| \leq |t| \quad \text{for} \ t \in I;
\]

since \( w' \) is increasing we find that

\[
|f'(t)| \leq (v(|t|) = \frac{|t|}{\sqrt{1 - t^2}} \quad \text{for} \ t \in I \tag{6}
\]

where \( v \) is the function inverse to \( w' \). This in turn implies that

\[
|f(t)| \leq 1 - \sqrt{1 - t^2} \quad \text{for} \ t \in I \tag{7}
\]

Let \( t_L = \inf I \) and let \( t_R = \sup I \). Owing to (6) we find that the limits

\[
x_L = \lim_{t \downarrow t_L} (t, f(t)) \quad \text{and} \quad x_R = \lim_{t \uparrow t_R} (t, f(t))
\]

exist and are in \( \text{spt} \partial[E] \). Owing to the regularity properties of \( \text{bdry} \ E \) and the estimate (6) we find that if either \(-1 < t_L \) or \( t_R < 1 \) the maximality of \( I \) is contradicted. \( \square \)

Theorem 3.1.2. Suppose \( C \in B_1(\mathbb{R}^2), C \) is compact and convex, \( a \in \text{bdry} \ C, \)
\( v \in S^1 \) and
\(-v \in \text{Nor}(C, a). \)

Then
\( B(a + v, 1) \subset C. \)

Proof. It will suffice to consider the case \( a = 0 \) and \( v = e_2 \).

Let
\[ a^- = \inf \{ x_1 : x \in C \} < 0, \quad a^+ = \sup \{ x_1 : x \in C \} > 0, \quad b = \sup \{ x_2 : x \in C \} > 0 \]

and let
\[ f^\pm : [a^-, a^+] \to [0, b] \]
be such that
\[ f^-(t) = \inf \{ u : (t, u) \in C \} \quad \text{and} \quad f^+(t) = \sup \{ u : (t, u) \in C \} \quad \text{for} \ a^- \leq t \leq a^+. \]
Then $f^-$ is convex, $f^+$ is concave, $f^- \leq f^+$ and

$$C = \{(x_1, x_2) \in [a^-, a^+] \times \mathbb{R} : f^-(x_1) \leq x_2 \leq f^+(x_1)\}.$$

Let $c$ be such that $(c, b) \in C$.

Applying Theorem 3.1.1 with $a = 0$, $u = e_1$ and $v = e_2$ we find that $a^- \leq -1$, that $1 \leq a^+$, that

$$|(t, f^-(t)) - (0, 1)| \geq 1 \quad \text{for} \quad a^- \leq t \leq a^+.$$

Applying Theorem 3.1.1 with $a = (a^+, f^-(a^+))$, $u = e_2$ and $v = -e_1$ we find that $f^-(a^+) \geq 1$, that $b \geq f^+(a^+) + 1 \geq 2$ and that

$$|(t, f^-(t)) - (a^+ - 1, f^-(a^+))| \geq 1 \quad \text{for} \quad a^- - 1 \leq t \leq a^+.$$

and that

$$|(t, f^+(t)) - (a^+ - 1, f^+(a^+))| \geq 1 \quad \text{for} \quad a^- - 1 \leq t \leq a^+.$$

Applying Theorem 3.1.1 with $a = (a^-, f^-(a^-))$, $u = e_2$ and $v = e_1$ we find that $f^-(a^-) \geq 1$, that $b \geq f^+(a^-) + 1 \geq 2$, that

$$|(t, f^-(t)) - (a^- + 1, f^-(a^-))| \geq 1 \quad \text{for} \quad a^- \leq t \leq a^- - 1.$$

and that

$$|(t, f^+(t)) - (a^- + 1, f^+(a^-))| \geq 1 \quad \text{for} \quad a^- \leq t \leq a^- - 1.$$

That the Theorem holds should now be clear.

\[ \square \]

### 3.2 Convex Chan-Esedoglu Theorem.

**Theorem 3.2.1.** Suppose

(i) $S$ is a compact convex subset of $\mathbb{R}^2$;

(ii) $0 < \epsilon < \infty$, $D \in n^l_{loc}(V_S)$ and $D =< D >$;

(iii) $T = \bigcup \{B(c, \epsilon) : c \in \mathbb{R}^2 \text{ and } B(c, \epsilon) \subset S\}$.

Then

$$D \neq \emptyset \Rightarrow D = T.$$

The Theorem will follow from the following ? Lemmas. We may suppose without loss of generality that $D$ is compact. Recall from ? that $D \subset S$.

**Lemma 3.2.1.** Suppose $a, b, c, A$ are such that $A \in \text{int}(D, S)$, $A \subset C(c, 1)$, $\text{ends}(A) = \{a, b\}$; and $V = \{t(x - c) : x \in \text{cl} A \text{ and } t > 1\}$.

Then $D \cap V = \emptyset$. 

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Proof. Suppose $D \cap V \neq \emptyset$. Then (ii) of Proposition ?? holds. Let $A', a', b', c'$ and $G$ be as in ???. Then $D \cap G = \emptyset$.

Since $G$ is a subset of the convex hull of $\{a, b, a', b'\}$ we infer that $G \subset S$ so $D \cup G \subset S$. Thus

$$U(D \cup G) - U(D) = L^2(G) = 2|c - c'| - \pi;$$

moreover,

$$\TV(D \cup G) - \TV(D) = \TV(D) - 2\pi + 2|c - c'| - \TV(D) = 2|d - d'| - 2\pi.$$

It follows that

$$0 \leq (\TV(D \cup G) + U(D \cup G) - (\TV(D) + U(D)) = -\pi < 0$$

which is a contradiction. \hfill \Box

Lemma 3.2.2. $D$ is convex.

Proof. Suppose $e \in \bdry D$.

If $e \in \bdry S$ then $D \subset S \subset e + \Tan(S, e)$.

Suppose $e \notin \bdry S$. Then $e \in \inter S$ so there is $A \in \inter(D, S)$ such that $e \in A$. Let $a, b, c$ be such that $A \subset C(c, 1)$ and $\text{ends}(A) = \{a, b\}$. Let $V = \{t(x - c) : x \in \cl A \text{ and } t > 1\}$. We have $D \cap V = \emptyset$ from the preceding Lemma. Since $S$ is a subset of the intersection of the closed halfspaces $a + \Tan(D, a)$ and $b + \Tan(D, b)$ and $D \subset S$ we conclude that $D \subset e + \Tan(C(c, 1), e)$.

It follows that $D$ is convex. \hfill \Box

Lemma 3.2.3. Suppose $u \in S^1$. Then

$$\sup\{x \cdot u : x \in S\} - \inf\{x \cdot u : x \in S\} > 2.$$

Proof. Let

$$b^- = \inf\{x \cdot u : x \in D\} \quad \text{and let} \quad b^+ = \sup\{x \cdot u : x \in D\}.$$

Suppose the Lemma were false. Then, as $D \subset S$, $b^+ - b^- \leq 2$. Let $v \in S^1$ be such that $u \cdot v = 0$. Let

$$a^- = \inf\{x \cdot v : x \in D\} \quad \text{and let} \quad a^+ = \sup\{x \cdot v : x \in D\}.$$

Then $L^2(D) \leq (a^+ - a^-)(b^+ - b^-) = 2(a^+ - a^-)$. Since $D$ is convex by the preceding Lemma we have that $\TV(D) > 2(a^+ - a^-)$. Thus

$$\TV(D) + U(D) = \TV(D) - L^2(D) > 0 = \TV(\emptyset) + U(\emptyset)$$

which is a contradiction. \hfill \Box

Lemma 3.2.4. $D \subset T$. 

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Proof. Suppose \( d \in D \). Let \( a \in \text{bdry } D \) be such that \( |d - a| = \text{dist}(d, \text{bdry } D) \). In case \( |d - a| \geq 1 \) we have \( d \in B(d, 1) \subset D \) so \( d \in T \).

Suppose \( |d - a| < 1 \). Letting \( v \in S^1 \) be such that \( v \in \text{Nor}(D, a) \) we have from Lemma 3.2.4 that \( B(a + v, 1) \subset D \). Since \( d \in B(a + v, 1) \) we have \( d \in T \). □

Lemma 3.2.5. \( T \subset D \).

Proof. Suppose, to the contrary, there were \( e \in T \sim D \). Let \( d \in \text{bdry } D \) be such that \( U(e, |d - e|) \cap D = \emptyset \). Then \( d - e \in \text{Nor}(D, d) \) so \( d \not\in d + \text{Tan}(D, d) \).

Were it the case that \( d \in \text{bdry } S \) we would have \( d \in S \subset d + \text{Tan}(D, d) \); thus \( d \in \text{int } S \).

Let \( A \in \text{int}(D, S) \) be such that \( d \in A \) and let \( a, b, c \) be such that \( A \subset C(c, 1) \) and \( \text{ends}(A) = \{a, b\} \). Let \( J \) be the intersection of the closed halfspaces \( a + \text{Tan}(D, a) \) and \( b + \text{Tan}(D, b) \). Since \( e \) belongs to a closed ball of radius 1 which is a subset of \( S \) and therefore a subset of \( J \) we infer that the length of \( A \) equals \( \pi \). Thus the lines \( a + \text{Tan}(\text{bdry } D, a) \) and \( b + \text{Tan}(\text{bdry } D, b) \) are parallel with distance 2 between them; this is excluded by Lemma 3.2.3. □

4 Two circles.

Suppose \( 1 \leq l < \infty \). Let \( c_\pm = (\pm l, 0) \), let \( S^- = B(c_-, 1) \) and let \( S = S^- \cup S^+ \).

Let \( \Sigma \) be the group of rigid motions which carry \( S \) into itself. Thus \( \Sigma \) has four elements: \( \rho_1, \rho_2 \) and \( \pm 1 \) times the identity map of \( \mathbb{R}^2 \).

4.1 Preliminaries.

For reason which will become clear shortly we let

\[
G = \left( \frac{l^2 - 1}{2}, \infty \right) = \{ s \in (0, \infty) : l < \sqrt{2s + 1} \},
\]

\[
H = \left\{ \beta \in (0, \pi/2) : \sin \beta < \frac{2l}{l^2 + 1} \right\} = \left\{ \beta \in (0, \pi/2) : l < \frac{1 + \cos \beta}{\sin \beta} \right\}.
\]

Letting \( b(s) = \arcsin \frac{l}{1 + s} \) for \( s \in G \) and \( \sigma(\beta) = \frac{l - \sin \beta}{\sin \beta} \) for \( \beta \in H \)

we find that \( b \) and \( \sigma \) are inverse to each other.

Proposition 4.1.1. \( s \in G \) if and only if \( 0 < s < \infty \) and there exists \( t > 0 \) such that \( C = C((0, t), s) \) meets the boundaries of \( S^+ \) and \( S^- \) tangentially and \( C \subset \{ x \in \mathbb{R}^2 : x_2 > 0 \} \).
Proof. Suppose \(0 < s < \infty, t > 0\), \(C = C((0, t), s)\), \(C\) meets the boundaries of \(S^+\) and \(S^-\) tangentially and \(C \subset \{x \in \mathbb{R}^2 : x_2 > 0\}\). Then \((0, t) - c_{\pm} = s + 1\) and \(t - s > 0\) so \((s + 1)^2 = t^2 + t^2 > t^2 + s^2\) so \(s \in G\).

On the other hand, if \(s \in G\), \(t = \sqrt{(s + 1)^2 - l^2}\) and \(C = C((0, t), s)\) then \(C\) meets the boundaries of \(S^+\) and \(S^-\) tangentially. Moreover, \(t > s\) so \(C \subset \{x \in \mathbb{R}^2 : x_2 > 0\}\).

\[\Box\]

**Definition 4.1.1.** For each \(s \in G\) we let

\[F_s\]

be the compact subset of \(\mathbb{R}^2\) whose boundary equals the union of the four arcs

\[A^+, A^-, (\text{bdry } S^+) \sim B^+, (\text{bdry } S^-) \sim B^-\]

where, with

\[\beta = b(s), \quad \gamma = \pi/2 - \beta, \quad e_{\pm} = (0, \pm l \cot \beta),\]

we have set

\[A^+ = a(e^+, s, 3\pi/2 - \beta, 3\pi/2 + \beta), \quad A^- = a(e^-, s, \pi/2 - \beta, \pi/2 + \beta)\]

and

\[B^+ = a(e^+, 1, \pi - \gamma, \pi + \gamma), \quad B^- = a(e^-, 1, -\gamma, \gamma).\]

**Proposition 4.1.2.** Suppose \(s \in G\) and \(\beta = b(s)\). Then

\[TV(F_s) = 2\pi + 4\beta(1 + s);\]
\[L^2(F_s \cap S) = 2\beta;\]
\[L^2(F_s \sim S) = 2l(1 + s) \cos \beta + 2\beta(1 - s^2) - \pi.\]

\[\text{(8)}\]

Proof. We adopt the notation of the preceding Definition. The lengths of \(A^+, A^-, \text{bdry } S^+ \sim B^+, \text{bdry } S^- \sim B^-\) are

\[2\beta s, \quad 2\beta s, \quad 2\pi - 2\gamma, \quad 2\pi - 2\gamma,\]

respectively; \(TV(F_s)\) is the sum of these lengths which is

\[4\beta s + 4\pi - 4(\pi/2 - \beta) = 2\pi + 4\beta(1 + s).\]

It is obvious that \(L^2(F_s \cap S) = 2\pi\).

Let \(P\) be the parallelogram with vertices \(c_{\pm}, e_{\pm}\), let

\[Q = A(e_-, s, 3\pi/2 - \beta, \pi/2 + \beta) \cup A(e_+, s, 3\pi/2 - \beta, 3\pi/2 + \beta)\]

and let

\[R = A(c_+, 1, \pi - \gamma, \pi + \gamma) \cup A(c_-, 1, -\gamma, \gamma).\]

Then

\[L^2(F_s \sim S) = L^2(P) - L^2(Q) - L^2(R)\]
\[= 2l(1 + s) \cos \beta - 2\beta s^2 - 2\gamma\]
\[= 2l(1 + s) \cos \beta + 2\beta(1 - s^2) - \pi.\]

\[\Box\]

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4.2 The main theorem.

We suppose throughout this subsection that \(0 < r < \infty, 0 < s < \infty\) and
\[E \in \Gamma(S, r, s)\].

**Lemma 4.2.1.** Suppose \(A \in \text{ext}(E, S)\). Then \(\text{ends}(A)\) meets both \(\text{bdry} \, S^-\) and \(\text{bdry} \, S^+\).

**Proof.** Suppose the Lemma were false. Replacing \(E\) by \(\rho_2[E]\) if necessary we may assume that \(\text{ends}(A) \subset \text{bdry} \, S^+\). Since the length of \(A\) is less than \(2\pi s\) the circle containing \(A\) meets \(\text{bdry} \, S^+\) transversely. Since \(E\) is a subset of the convex hull of \(S\) we infer that \(\text{ends}(A) \subset \{x \in \mathbb{R}^2 : x_1 < l\}\).

Let \(m\) be the midpoint of \(A\). Replacing \(E\) by \(\rho_1[E]\) if necessary we may assume that \(m_2 \geq 0\). Let \(a, b \in \text{ends}(A)\) be such that \(a_2 < b_2\). Then
\[b_2 = \frac{b_2 + a_2}{2} + \frac{b_2 - a_2}{2} > \frac{b_2 + a_2}{2} = m_2.\]

Let \(Y = \text{bdry} \, S^+\) and let \(G\) be as in Theorem 2.2.1 (iii). Let \(X\) be the connected component of \(m\) in \(G \cap \text{bdry} \, E\), note that 2.2.1 (iv) holds, and let \(P, Q, R, \text{et cetera}\), be as in Theorem 2.2.1. Since \(\text{ends}(A) \subset Q\) we find that (II) or (III) of Theorem 2.2.1 hold. It follows from (\(\Gamma_0\)) and (\(\Gamma_1\)) that
\[(1) \quad Q \subset \{x \in \mathbb{R}^2 : x_1 < l\}.\]

Let \(N\) be a positive integer and let
\[q^0, q^1, \ldots, q^N\]
be the points of \(Q\) ordered in the clockwise way around the circle \(\text{bdry} \, S^+\) with \(q^0 = a\).

If (III) of Theorem 2.2.1 there would be points of \(Q\) in \(\{x \in \mathbb{R}^2 : x_1 \geq l\}\) which we have excluded. Thus (II) of Theorem 2.2.1 holds and this implies that there is \(A \in \mathcal{R}_3\) such that \(\text{ends}(A)\) contains \(q^N\) and meets the complement of \(X\) which is a subset of \(\text{bdry} \, S^−\). But this is impossible since, by Theorem 2.2.1 (xiv), \(A \subset X \subset \mathcal{B}(c_+, |m - c_+|)\). \(\Box\)

**Lemma 4.2.2.** Suppose
\[\rho_1[\text{Tan}(E, e)] = \text{Tan}(E, e)\quad \text{whenever } e \in L_1 \cap \text{bdry} \, E,\] \(A \in \text{ext}(E, S)\) and \(C\) is the circle containing \(A\). Then \(s \in G\) and Then
\[C = C(\pm te_2, s) \quad \text{where} \quad t = \sqrt{(r + s)^2 - l^2}.\]

**Remark 4.2.1.** By example one finds that the Lemma is false without the symmetry assumption.
Proof. Since \textbf{ends}(A) meets both \textbf{bdry} S− and \textbf{bdry} S+ there is e such that \(L_2 \cap A = \{e\}\). Let \(T = \text{Tan}(\text{bdry} E, e)\). Since \(\rho_1(e) = e\) we infer from Lemma 4.3.1 that \(\rho_1[T] = T\). Thus \(T = L_1\) or \(T = L_2\). From \((\Gamma 0)\) and \((\Gamma 5)\) we infer that \(T = L_1\). Let \(c\) be the center of the circle \(C\) containing \(A\) and note that \(c \in L_2\).

If \(C\) meets \textbf{bdry} S+ tangentially the Lemma holds, so suppose it does not. Let \(b\) be such that \textbf{ends}(A) ∩ \textbf{bdry} S+ = \{b\} and let \(B \in \text{int}(E, S)\) be such that \(b = \text{ends}(A) \cap \text{ends}(B)\). Let \(b'\) be such that \textbf{ends}(B) = \{b, b'\}. Since the length of \(B\) is less than \(2\pi r\) the circle containing \(B\) meets \textbf{bdry} S+ transversely so there is \(A' \in \text{ext}(E, S)\) such that \textbf{ends}(A') ∩ \textbf{ends}(B) = \{b'\}. By the preceding Lemma there is \(\{e'\} = L_2 \cap A'\). Let \(c'\) be the center of the circle containing \(A'\) and note that \(c' \in L_2\) for the same reasons that \(c \in L_2\). The center of the circle containing \(B\) is the point of intersection of the line containing \(b\) and \(c\) with the line containing \(b'\) and \(c'\). This forces the length of \(A\) to exceed \(\pi r\).

\(\square\)

**Theorem 4.2.1.** Suppose \(\rho_1[E] = E\) and \(S \notin \{\emptyset, S^+, S^-, S^- \cup S^+\}\).

Then \(s \in G\) and \(E = F_s\).

Proof. If \(E\) meets the complement of \(S^- \cup S^+\) then \(E = F_s\) by virtue of the preceding Lemma. If \(E \subset S^- \cup S^+\) it a straightforward matter which we leave to the reader to verify that \(S \in \{\emptyset, S^+, S^-, S^- \cup S^+\}\); the point here is that any member of \(\text{int}(E, S)\) must meet the boundary of \textbf{bdry} S− or \textbf{bdry} S+ transversely.

\(\square\)

### 4.3 The Chan-Esedoglu functional.

Suppose

\[0 < \epsilon < \infty, \quad E \in n_\epsilon(V_S) \quad \text{and} \quad E = \langle E \rangle.\]

We shall determine \(E\).

From \(\langle E \rangle\) we have

\[E \in \Gamma(S, \epsilon, \epsilon).\]

We have

\[
\begin{align*}
(V_S)_\epsilon(\emptyset) &= 0, \\
(V_S)_\epsilon(S^-) &= (V_S)_\epsilon(S^+) = \pi(2\epsilon - 1), \\
(V_S)_\epsilon(S^+) &= 2\pi(2\epsilon - 1)
\end{align*}
\]

(10)

#### 4.3.1 The case \(\epsilon \notin G\).

Suppose \(\epsilon \notin G\). By Theorem 4.2.1,

\[E \in \{\emptyset, S^-, S^+\}.\]

From (10) we find that

\[E = \begin{cases} \\
\emptyset & \text{if } \epsilon > 1/2, \\
\emptyset \text{ or } S^- \text{ or } S^+ & \text{if } \epsilon = 1/2, \\
S^- \text{ or } S^+ & \text{if } \epsilon < 1/2.
\end{cases}\]
4.3.2 The case $\epsilon \in G$.
So let us suppose $\epsilon \in G$.

Lemma 4.3.1. Suppose

(i) $S$ is a compact subset of $\mathbb{R}^2$ and $\zeta$ is a rigid motion of $\mathbb{R}^2$ such that $g[S] = S$;

(ii) $E \in n^l_{\epsilon}(N_S)$ and $E$ is compact;

(iii) $e \in \text{bdry } E$ and $\zeta(e) = e$.

Then

$$\zeta[\text{Tan}(E, e)] = \text{Tan}(E, e).$$

Proof. Suppose, contrary to the Lemma, $\zeta[\text{Tan}(E, e)] \neq \text{Tan}(E, e)$. By Theorem ?? in [CE] $D = E \cup g[E] \in n^l_{\epsilon}(N_S)$, But $\text{Tan}(D, e) = \text{Tan}(E, e) \cup g[\text{Tan}(E, e)]$ which is not a half space, contradicting the Regularity Theorem [AW, ??].

By Theorem 4.2.1,

$E \in \{F, \emptyset, S^-, S^+, S\}.$

4.3.3 $\Phi$ and $\Psi$.

For $(K, e) \in \mathbb{R}$ we set

$$\Phi(K, e) = e(2\pi + 4(\arcsin K)(1 + e))$$

$$+ 2K(1 + e)\sqrt{1 - K^2} + 2(\arcsin K)(1 - e^2) - \pi - 2\pi$$

and we set

$$\Psi(K, e) = 4\pi e - 2\pi.$$

Proposition 4.3.1. There is one and only one function

$$P : (0, 1) \to (0, \infty)$$

such that if $(K, e) \in (0, 1) \times (0, \infty)$ then

$$\Phi(K, e) \iff e = P(K).$$

Moreover, $P$ is decreasing.

Proposition 4.3.2. There is one and only one function

$$Q : (0, 1) \to (0, \infty)$$

such that if $(K, e) \in (0, 1) \times (0, \infty)$ then

$$\Phi(K, e) = \Psi(K, e) \iff e = Q(K).$$
Moreover, $Q$ is decreasing and there is one and only one $K^* \in (0,1)$ such that if $0 < K < 1$ then
\[
\frac{1}{2} < P(K) < Q(K) \iff K < K^*; \\
\frac{1}{2} = P(K) = Q(K) \iff K = K^*; \\
\frac{1}{2} < P(K) > Q(K) \iff K > K^*
\]
Finally,
\[
\{(K,e) : e \leq Q(K)\} \subset J.
\]

\[
R_\emptyset = \{(K,e) : 0 < K < K^* \text{ and } e > P(K)\} \\
\quad \cup \{(K,e) : K = K^* \text{ and } \frac{1}{2} < e\} \\
\quad \cup \{(K,e) : K^* < K < 1 \text{ and } \frac{1}{2} < e\};
\]
\[
R_{\emptyset,S} = \{(K,e) : K^* < K < 1 \text{ and } e = \frac{1}{2}\};
\]
\[
R_{\emptyset,F} = \{(K,e) : 0 < K < K^* \text{ and } e = P(K)\};
\]
\[
R_{\emptyset,S,F} = \left\{ \left( K^*, \frac{1}{2} \right) \right\};
\]
\[
R_S = \{(K,e) : K^* < K < 1 \text{ and } Q(K) < e < \frac{1}{2}\};
\]
\[
R_{S,F} = \{(K,e) : K^* < K < 1 \text{ and } e = Q(K)\};
\]
\[
R_{F} = \{(K,e) : 0 < K < K^* \text{ and } e < P(K)\} \\
\quad \cup \{(K,e) : K = K^* \text{ and } e < \frac{1}{2}\} \\
\quad \cup \{(K,e) : K^* < K < 1 \text{ and } e < Q(K)\};
\]

### 4.3.4 The gory details.

Let
\[
R = \{(K,e) : 0 < K < 1 \text{ and } 0 < e < \infty, \}
\]
let
\[
j(K) = \frac{1 - K^2 + \sqrt{1 - K^2}}{K^2} \text{ for } 0 < K < 1
\]
and let
\[
J = \{(K,e) \in (0,1) \times (0,\infty) : e < j(K)\}.
\]
Note that
\[ J = \left\{ (K, e) \in (0, 1) \times (0, \infty) : \frac{(K(1+e))^2 - 1}{2} < e \right\} \]
\[ = \left\{ (K, e) \in (0, 1) \times (0, \infty) : K < \frac{\sqrt{2e+1}}{1+e} \right\} \]

For \((K, e) \in R\) we set
\[ \Phi(K, e) = e(2\pi + 4(\arcsin K)(1 + e)) + 2K(1 + e)\sqrt{1 - K^2} + 2(\arcsin K)(1 - e^2) - \pi \]
\[ - 2\pi \]
and we set
\[ \Psi(K, e) = 4\pi e - 2\pi. \] (12)

For \(0 < K < 1\) we set
\[ \Phi_2(K) = 2(\arcsin K + K\sqrt{1 - K^2}), \]
\[ \Phi_1(K) = 2(\pi + 2\arcsin K + 2K\sqrt{1 - K^2}), \]
\[ \Phi_0(K) = 2K\sqrt{1 - K^2} + 2\arcsin K - 3\pi \]

We set
\[ \Phi(K, e) = \Phi_2(K)e^2 + \Phi_1(K)e + \Phi_0(K) \quad \text{and} \quad \Psi(K, e) = 4\pi e - 2\pi \]
for \((K, e) \in (0, 1) \times (0, \infty).\)

With
\[ e = s = \epsilon, \quad \beta = \arcsin \frac{l}{1+\epsilon}, \quad K = \frac{l}{1+\epsilon} \]
we find that
\[ (K, e) \in J \]
and we infer from Proposition 4.1.2 that
\[ M_\epsilon(F_\epsilon) = \epsilon(2\pi + 4\beta)(1 + s) + 2l(1 + s)\cos\beta + 2\beta(1 - s^2) - \pi \]
\[ - 2\pi \]
\[ = \Phi(K, e); \]
(13)
also,
\[ M_\epsilon(S) = \Psi(K, e). \] (14)

Also,
\[ (K, e) \in J \text{ and } l = K(1 + e) \implies e \in J. \]
For $0 < K < 1$ we set
\[
\Phi_2(K) = 2(\arcsin K + K\sqrt{1 - K^2}), \\
\Phi_1(K) = 2(\pi + 2\arcsin K + 2K\sqrt{1 - K^2}), \\
\Phi_0(K) = 2\sqrt{1 - K^2} + 2\arcsin K - 3\pi
\]
and note that
\[
\Phi(K, e) = \Phi_2(K)e^2 + \Phi_1(K)e + \Phi_0(K) \quad \text{for } (K, e) \in (0, 1) \times (0, \infty).
\]

4.4 Analysis of $\Phi$ and $\Psi$.

We have
\[
\Phi_2 > 0, \quad \Phi_1 > 0, \quad \Phi_0 < 0.
\]
It follows that whenever $0 < K < 1$ there is one and only one positive $P(K)$ such that
\[
\Phi(K, P(K)) = 0.
\]
Since $\Phi_2 > 0$ we find that, for any $(K, e) \in R$,\begin{align*}
\Phi(K, e) < 0 & \iff e < P(K), \\
\Phi(K, e) = 0 & \iff e = P(K), \\
\Phi(K, e) > 0 & \iff e > P(K).
\end{align*}
(15)
For any $(K, e) \in R$ we have\begin{align*}
\frac{\partial \Phi}{\partial e}(K, e) &= 2(\pi + 2(1 + e)(\arcsin K + K(1 - K^2)^{1/2})) > 0, \\
\frac{\partial \Phi}{\partial K}(K, e) &= \frac{4}{(1 - K^2)^{1/2}}(1 - K)^2(1 + e)^2 > 0.
\end{align*}
(16)
It follows that
\[
(0, 1) \ni K \mapsto P(K) \quad \text{is decreasing}.
\]
Since
\[
\lim_{K \uparrow 1} P(K) = -2 + \sqrt{6} < \frac{1}{2} < \frac{3}{2} = \lim_{K \downarrow 0} P(K)
\]
we find there is a unique
\[
K^* \in (0, 1) \quad \text{such that } P(K^*) = \frac{1}{2}.
\]
Using \texttt{fsolve} in Maple we obtain
\[
K^* \approx 0.7908979175.
\]
Since the line $e = 1/2$ crosses the boundary of $J$ at
\[
\left( \frac{2\sqrt{3}}{2}, \frac{1}{2} \right)
\]
and since
\[
\frac{2\sqrt{3}}{2} \approx 0.9428090414
\]
we find that
\[
\left( K^*, \frac{1}{2} \right) \in J.
\]

For any $(K, e) \in J$ it is obvious that
\[
\Psi(K, e) < 0 \iff e < \frac{1}{2},
\]
\[
\Psi(K, e) = 0 \iff e = \frac{1}{2},
\]
\[
\Psi(K, e) > 0 \iff e > \frac{1}{2}.
\]

(17)

For $(K, e) \in R$ we have
\[
\Phi(K, e) - \Psi(K, e) = \Phi_2(K) e^2 + (\Phi_1(K) - 4\pi)e + (\Phi_0 + 2\pi)(K).
\]

Note that
\[
\Phi_2 > 0, \quad \Phi_1 - 4\pi < 0, \quad \Phi_0 + 2\pi < 0.
\]

It follows that whenever $0 < K < 1$ there is one and only one positive $Q(K)$ such that
\[
\Phi(K, Q(K)) = \Psi(K, Q(K)).
\]

Since $\Phi_2 > 0$ we find that, for any $(K, e) \in R$,
\[
\Phi(K, e) < \Psi(K, e) \iff e < Q(K),
\]
\[
\Phi(K, e) = \Psi(K, e) \iff e = Q(K),
\]
\[
\Phi(K, e) > \Psi(K, e) \iff e > Q(K).
\]

(18)

Since
\[
\frac{1}{2} ((\Phi_1 - 4\pi)/\Phi_2)'(K) = ((\Phi_0 + 2\pi)/\Phi_2)'(K) = -\frac{(1 - K^2)^{1/2}}{(\arcsin K + K\sqrt{1 - K^2})^2} > 0
\]
for $0 < K < 1$ we infer that
\[
(0, 1) \ni K \mapsto Q(K) \quad \text{is decreasing.}
\]

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Suppose $0 < K < 1$. We have

$$\Phi(K, P(K)) - \Psi(K, P(K)) = 0 - 2\pi(2P(K) - 1) \begin{cases} < 0 & \text{if } P(K) > 1/2, \\ = 0 & \text{if } P(K) = 1/2, \\ > 0 & \text{if } P(K) < 1/2. \end{cases}$$

Since $P$ is decreasing and $P(K^*) = 1/2$ it follows that

$$P(K) < Q(K) \iff K < K^*;$$

$$P(K) = Q(K) \iff K = K^*;$$

$$P(K) > Q(K) \iff K > K^*.$$ 

(19)

We let

$$v(K, e) = \min\{0, \Phi(K, e), \Psi(K, e)\} \text{ for } (K, e) \in J.$$ 

We let

$$V_{0,\Phi,\Psi} = \{v = 0\} \cap \{v = \Phi\} \cap \{v = \Psi\};$$

$$V_{0,\Phi} = \{v = 0\} \cap \{v = \Phi\} \cap \{v < \Psi\};$$

$$V_{0,\Psi} = \{v = 0\} \cap \{v < \Phi\} \cap \{v = \Psi\};$$

$$V_0 = \{v = 0\} \cap \{v < \Phi\} \cap \{v < \Psi\};$$

$$V_{\Phi,\Psi} = \{v < 0\} \cap \{v = \Phi\} \cap \{v = \Psi\};$$

$$V_{\Phi} = \{v < 0\} \cap \{v = \Phi\} \cap \{v < \Psi\};$$

$$V_{\Psi} = \{v < 0\} \cap \{v < \Phi\} \cap \{v = \Psi\}.$$ 

Proposition 4.4.1. We have

$$V_{0,\Phi,\Psi} = \{(K^*, 1/2)\};$$

$$V_{0,\Phi} = \{(K, e) \in J : P(K) = e \text{ and } e > 1/2\};$$

$$V_{0,\Psi} = \{(K, e) \in J : K > K^* \text{ and } e = 1/2\};$$

$$V_{\Phi,\Psi} = \{(K, e) \in J : K > K^* \text{ and } e = Q(K)\};$$

$$V_0 = \{(K, e) \in J : P(K) < e \text{ and } 1/2 < e\};$$

$$V_{\Phi} = \{(K, e) \in J : e < P(K) \text{ and } K \leq K^*\} \cup \{(K, e) \in J : e < Q(K) \text{ and } K > K^*\};$$

$$V_{\Psi} = \{(K, e) \in J : K > K^* \text{ and } Q(K) < e < 1/2\};$$

Moreover, none of these sets are empty.

We need to relate all this to $J$.

For this purpose, let

$$S(K) = \Phi(K, j(K)) \text{ for } 0 < K < 1.$$
Let
\[ S_1 = (4K + \pi K^2)(1 - K^2) + \pi K^2; \]
\[ S_2 = 2K(2 + \pi K - K^2); \]
\[ S_3 = 8 - 6K^2; \]
\[ S_4 = 8 - 2K^2. \]

Then
\[ S'(K) = -2 \frac{S_1(K) + S_2(K)(1 - K^2)^{1/2} + (S_3(K) + (1 - K^2)^{1/2}S_4(K)) \arcsin(K)}{K^3(1 - K^2)^{1/2}} < 0 \]
so \( S \) is decreasing. Since
\[ \lim_{K \to 0} S(K) = \infty \quad \text{and} \quad \lim_{K \to 1} S(K) = -2\pi \]
we infer that \( S(K) = 0 \) for exactly one \( K \) with \( 0 < K < 1 \). Since \( \lim_{K \to 0} P(K) = 3/2 < \infty \) and \( \lim_{K \to 1} P(K) = -2 + \sqrt{6} > 0 \) we infer the graph of \( P \) intersects the graph of \( j \) in exactly one point. Using \texttt{fsolve} in Maple we find that this point is approximately
\((0.9498043393, 0.4552748006)\).

5 The ROF functional.

Suppose
\[ 0 < \epsilon < \infty. \]

Let
\[ F(g) = \frac{1}{2} \int_{\mathbb{R}^2} |g - 1| d\mathcal{L}^2 \quad \text{for} \ g \in F(\mathbb{R}^2). \]

Let
\[ Y^* = \begin{cases} \frac{2\epsilon}{l^2 - 1} & \text{if } l > 1, \\ \infty & \text{if } l = 1. \end{cases} \]

Suppose
\[ f \in m_\epsilon(F). \]

Note that
\[ 0 \leq f(x) \leq 1 \quad \text{for almost all } x \in \mathbb{R}^2. \]

We let
\[ E_y = \text{cl} \{ f \geq y \} \quad \text{for } 0 < y < 1. \]

Since \( F \) and therefore \( F_\epsilon \) is strictly convex we have
\[ [f \circ \sigma] = [f] \quad \text{for } \sigma \in \Sigma \]
which implies that
\[ \sigma[E_y] = E_y \quad \text{for } 0 < y < 1 \text{ and } \sigma \in \Sigma. \]

Whenever \( 0 < y < 1 \) we let
\[ U_y(D) = y\mathcal{L}^2(S \sim D) - (1 - y)\mathcal{L}^2(D \cap S) \quad \text{for } D \in \mathcal{M}(\mathbb{R}^2) \]
and note that
\[ E_y \in \Gamma \left( S, \frac{\epsilon}{1 - y}, \frac{\epsilon}{y} \right) \]
so that, by Theorem 4.2.1 and ??,
\[ E_y \in \{S, \emptyset, F_{\epsilon/y}\} \cap n_\epsilon(U_y). \]

Let
\[
Y_1 = \{y \in (0, 1) : E_y = F_{\epsilon/y}\}; \\
Y_2 = \{y \in (0, 1) : E_y = S\}; \\
Y_3 = \{y \in (0, 1) : E_y = \emptyset\}.
\]

Since
\[ \emptyset \subset S \subset F_{\epsilon/y} \]
whenever \( 0 < y < 1 \) we find that of \( 1 \leq i < j \leq 3 \) then
\[ y_i < y_j \quad \text{if } y_i \in Y_i \text{ and } y_j \in Y_j. \]

It follows that \( Y_i, i = 1, 2, 3 \) is an interval.

We shall determine \( E_y, 0 < y < 1. \)

Note that
\[ (U_y)_\epsilon(S) = \epsilon(4\pi) - (1 - y)2\pi = 2\pi(2\epsilon - (1 - y)). \tag{20} \]

5.1
Suppose \( 0 < y < 1. \) Let
\[ s = \frac{\epsilon}{y}. \]
Then
\[ s \in G \iff \frac{\epsilon}{y} \geq \frac{l^2 - 1}{2}. \]

5.2 Case One.
Suppose
\[ \frac{\epsilon}{y} < \frac{l^2 - 1}{2}. \]
Then \( s \notin G. \) Then
\[
\begin{align*}
y > 1 - 2\epsilon & \implies E_y = \emptyset; \\
y = 1 - 2\epsilon & \implies E_y \in \{S, \emptyset\}; \\
y < 1 - 2\epsilon & \implies E_y = S.
\end{align*}
\]
5.3 Case Two.
Suppose \( \frac{\epsilon}{y} \geq \frac{l^2 - 1}{2} \).
Then \( s \in G \).

5.3.1 Some preliminary calculations.
Let 
\[ R = \{(B, \lambda) : 0 < B < 1 \text{ and } 1 \leq \lambda < \infty \} \]
let 
\[ \Lambda(B) = \frac{1 + (1 - B^2)^{1/2}}{B} \text{ for } 0 < B < 1 \]
and let 
\[ J = \{(B, \lambda) \in R : \lambda < \Lambda(B)\} \]
Evidently, 
\[ J = \{(B, \lambda) \in R : B < \frac{2\lambda}{\lambda^2 + 1}\} \]
Suppose \((B, l) \in R \) and \( s = \frac{l - B}{B} \). Then 
\[ (B, l) \in J \iff s \in G \]

For \((B, \lambda) \in R \) we let 
\[ \Phi(B, \lambda) = \epsilon(2\pi + 4B \left( 1 + \left( \frac{\lambda - B}{B} \right) \right)) \]
\[ - \left( 1 - \left( \frac{\epsilon B}{\lambda - B} \right) \right)(2\pi) \]
\[ + \left( \frac{\epsilon B}{\lambda - B} \right)(2\lambda(1 + \left( \frac{\lambda - B}{B} \right))(1 - B^2)^{1/2} + 2B \left( 1 - \left( \frac{\lambda - B}{B} \right)^2 \right) - \pi) \]
and 
\[ \Psi(B, \lambda) = 2\pi \left( 2\epsilon - \left( 1 - \frac{\epsilon B}{\lambda - B} \right) \right) . \]
for \((\lambda, B) \in J \).
We have 
\[ \Phi(\lambda, B) = \frac{\epsilon(2\pi \lambda B - \pi B^3 + 2\lambda^2 \arcsin B + 2\lambda^2 B\sqrt{1 - B^2} - 2\pi \lambda B + 2\pi B^2}{B(\lambda - B)} . \]
Suppose 
\[ 0 < y < Y^* \text{ so } s = \frac{\epsilon}{y} \in G . \]
If $\beta, B, \lambda$ are such that

$$B = \beta = \arcsin\left(\frac{l}{1 + s}\right), \quad \lambda = l$$

we find that

$$s = \frac{B}{\lambda - B}, \quad y = \frac{\epsilon B}{\lambda - B}, \quad (B, \lambda) \in J,$$

$$(U_y)_{\epsilon}(F/\epsilon) = e(2\pi + 4\beta(1 + s)) - (1 - y)(2\pi) + y(2l(1 + s)\cos \beta + 2\beta(1 - s^2) - \pi) = \Phi(B, \lambda)$$

and that

$$(U_y)_{\epsilon}(S) = \Psi(B, \lambda).$$

We have

$$\Phi(\lambda, B) = \frac{e(2\pi \lambda B - \pi B^2 + 2\lambda^2 \arcsin B + 2\lambda^2 B\sqrt{1 - B^2}) - 2\pi \lambda B + 2\pi B^2}{B(\lambda - B)}$$

for $(\lambda, B) \in J$.

**5.4 Analysis of $\Phi$ and $\Psi$.**

**Proposition 5.4.1.** We have

(i) $\frac{\partial \Phi}{\partial B}(B, \lambda) > 0$ for $(B, \lambda) \in J$;

(ii) $\lim_{B \to 0} \Phi(B, \lambda) = 2((2\lambda + \pi)\epsilon - \pi)$;

(iii) $\frac{\partial \Psi}{\partial B}(B, \lambda) > 0$ for $(B, \lambda) \in R$;

(iv) $\lim_{B \to 0} \Psi(B, \lambda) = 2\pi(2\epsilon - 1)$.

**Proof.** For $0 < B < 1$ we set

$$N_2(B) = -2((1 - B^2)^2 \arcsin(B) - B(1 - B^2)),$$

$$N_1(B) = 4B(1 - B^2)^2 \arcsin(B),$$

$$N_0(B) = \pi B^2(1 - B^2)^{1/2}$$

and let

$$N(B, \lambda) = N_2(B)\lambda^2 + N_1(B)\lambda + N_0(B) \quad \text{for } \lambda \in \mathbb{R} \text{ and } 0 < B < 1.$$ 

We have

$$\frac{\partial \Phi}{\partial B}(B, \lambda) = \frac{\lambda \epsilon}{B^2(\lambda - B)^2(1 - B^2)^{1/2}}N(B, \lambda) \quad \text{for } (B, \lambda) \in J.$$
Suppose \((B, \lambda) \in J\). Since \(N_2(B) < 0\) and \(\lambda \leq \Lambda(B)\) we have

\[
N(B) \geq N_2(\lambda)\Lambda(B)\lambda + N_1(B)\lambda + N_0(B)
\]

\[
= \frac{O_1(B)\lambda + O_0(B, \lambda)}{B}
\]

where we have set

\[
O_1(B) = -2 \arcsin(B) \sqrt{1 - B^2} - 2 \arcsin(B)
+ 2 \arcsin(B) B^2 + 4 \arcsin(B) \sqrt{1 - B^2}B^2 + 2 B
+ 2 \sqrt{1 - B^2}B - 2 B^3 - 2 B^3 \sqrt{1 - B^2}
\]

and

\[
O_0(B) = \sqrt{1 - B^2}B^3\pi.
\]

Since both \(O_1(B)\) and \(O_0(B)\) are positive, (i) holds.

Since

\[
\frac{\partial \Psi}{\partial B}(B, \lambda) = \frac{2\pi \epsilon \lambda}{(\lambda - B)^2}
\]

for \((B, \lambda) \in J\) we find that (iii) holds.

(iv) is obvious.

For \((B, \lambda) \in R\) we have

\[
(\Phi - \Psi)(B, \lambda) = \frac{\epsilon}{B(\lambda - B)} Z(B, \lambda)
\]

where for \((B, \lambda) \in J\) we have set

\[
Z_2(B) = 2 \left( \arcsin(B) + B(1 - B^2)^{1/2} \right),
\]

\[
Z_1(B) = -2\pi B,
\]

\[
Z_0(B) = \pi B^2
\]

and

\[
Z(B, \lambda) = Z_2(B)\lambda^2 + Z_1(B)\lambda + Z_0(B).
\]

For \(B \in (0, 1)\) we have

\[
\left( \frac{Z_0}{Z_2} \right)'(B) = \frac{B \arcsin B}{(\arcsin(B) + B(1 - B^2)^{1/2})^2} > 0
\]

and

\[
\left( \frac{Z_1}{Z_2} \right)'(B) = \frac{B - (1 - B^2)^{1/2} \arcsin B - B^3}{(\arcsin(B) + B(1 - B^2)^{1/2})(1 - B^2)^{1/2}} > 0.
\]

By 1.6 the quadratic equation \(Z(B, \lambda) = 0\) in \(\lambda\) has for each \(0 < B < 1\) has two solutions

\[
P^-(B) < P^+(B)
\]
such that $P^-$ is increasing and $P^+$ is decreasing. Evidently,

$$\lim_{B \to 0} P^-(B) = 0, \quad \lim_{B \to 1} P^-(B) = 1, \quad \lim_{B \to 0} P^+(B) = \frac{\pi}{2}, \quad \lim_{B \to 1} P^-(B) = 1.$$ 

Let

$$Q : \left(1, \frac{\pi}{2}\right) \to (0, 1)$$

be the function which is inverse to $P^+$. We have

**Theorem 5.4.1.** We have

$$\{(B, \lambda) \in R : \Phi(B, \lambda) = \Psi(B, \lambda)\} = \{(B, P^+(B)) : 0 < B < 1\} = \{(\lambda, Q(\lambda) : 1 < \lambda < \frac{\pi}{2}\}.$$ 

Now let

$$\Upsilon(B) = \Psi(B, P(B)) \quad \text{for } 0 < B < 1.$$ 

For any $(B, \lambda) \in R$ we have

$$\Psi(B, \lambda) = 2\pi(2\epsilon - 1) + 2\pi\epsilon \frac{B}{\lambda - B}$$

and

$$\frac{\partial}{\partial B} \frac{B}{\lambda - B} = \frac{\lambda}{(\lambda - B)^2}, \quad \frac{\partial}{\partial \lambda} \frac{B}{\lambda - B} = -\frac{B}{(\lambda - B)^2}.$$ 

Since $(P^+)' < 0$ we find that

$$\Upsilon'(B) > 0 \quad \text{for } 0 < B < 1.$$ 

Since

$$\lim_{B \to 0} \Upsilon(B) = 2\pi(2\epsilon - 1) \quad \text{and} \quad \lim_{B \to 1} \Upsilon(B) = \infty$$

**5.4.1**

We consider the following four cases:

(I) $(2l + \pi)\epsilon \geq \pi$ and $\epsilon \geq 1/2$;

(II) $(2l + \pi)\epsilon \geq \pi$ and $\epsilon < 1/2$;

(III) $(2l + \pi)\epsilon < \pi$ and $\epsilon \geq 1/2$;

(IV) $(2l + \pi)\epsilon < \pi$ and $\epsilon < 1/2$.

**In case (I) holds** we infer from ?? that $\Phi(B, l) > 0$ whenever $(B, l) \in J$ and that $\Psi(B, l) > 0$ whenever $(B, l) \in R$. Thus $Y_1$ and $Y_2$ are empty and $Y_3 = (0, \infty)$.

**In case (II) holds** we infer from ?? that $\Phi(B, l) > 0$ whenever $(B, l) \in J$. This implies $Y_1$ is empty. Since $\lim_{B \to 0} \Psi(B, l) < 0$ by ??, since $(0, 1) \ni B \mapsto \Psi(B, l)$ is increasing by ?? we find that $Y_2 = (0, 1 - 2\epsilon)$. 

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In case (III) holds we have
\[ 1 \leq l < \frac{\pi}{2}(1 - \epsilon) \leq \frac{\pi}{4} \]
which is impossible so this case is void.

In case (IV) holds we have
\[ \lim_{B \downarrow 0} \Upsilon(B) < 0 \quad \text{and} \quad \lim_{B \downarrow 0} \Upsilon(B) = \infty \]
so there is one and only one \( B^* \) such that
(i) \( \Upsilon(B) < 0 \iff B < B^* \);
(ii) \( \Upsilon(B) = 0 \iff B = B^* \);
(iii) \( \Upsilon(B) > 0 \iff B > B^* \)
whenever \( 0 < B < 1 \). Let
\[ \lambda^* = Q(B^*) \]
and note that \( 1 < \lambda^* < \infty \).

In case \( 1 \leq l \leq \lambda^* \) we have
\[ \Psi(B^*, l) = \Psi(B^*, l) \Upsilon(B^*) \geq 0. \]
Since \( (0, B^*) \ni B \mapsto \Phi(B, l) \) is increasing there and is one and only one \( B^{**} \in (0, B^*) \) such that
\[ \Psi(B^{**}, l) = 0. \]
We have
\[ \Psi(B, l) < \Psi(B, l) \quad \text{whenever} \quad 0 < B < B^* \]
and
\[ 0 \leq \Psi(B^*, l) < \Psi(B, l) \quad \text{whenever} \quad B^* < B < 1. \]
It follows that \( Y_2 = \emptyset \) and that \( Y_1 = (0, y^{**}) \) where
\[ y^{**} = \epsilon \frac{B^{**}}{1 - B^{**}}. \]

In case \( \lambda^* < l \) we have
\[ \Psi(B^*, l) = \Psi(B^*, l) \Upsilon(B^*) = 0. \]
We have
\[ \Phi(B, l) < \Psi(B, l) \quad \text{whenever} \quad 0 < B < B^*, \]
\[ \Phi(B^*, l) = \Psi(B^*, l), \]
and
\[ \Phi(B, l) > \Psi(B, l) \quad \text{whenever} \quad B^* < B < 1. \]
It follows that
\[ Y_1 = (0, y^*) \quad \text{and} \quad Y_2 = [y^*, 1 - 2\epsilon) \]
where
\[ y^* = \epsilon \frac{B^*}{l - B^*}. \]
6 Two squares.

Let
\[ T = [0, 1] \times [0, -1], \quad U = [-1, 0] \times [0, 1], \quad S = T \cup U. \]

Let \( \Sigma \) be the group of rigid motions which carry \( S \) into itself; \( \Sigma \) has four elements: \( \sigma \pm \) and \( \pm 1 \) times the identity map of \( \mathbb{R}^2 \).

Let
\[ V = \{(0, 0), (1, 0), (1, -1), (0, -1), (0, 1), (-1, 1), (-1, 0)\}; \]
thus \( V \) is the set of the seven vertices of the polygon \( S \).

Let
\[ \{(0, 0), (1, 0), (1, -1), (0, 1), (-1, 1), (-1, 0)\} \]
and let
\[ \{((0, 1), (-1, 1)), ((1, -1), (-1, 0)), ((-1, 0), (0, 1)), ((0, 0), (0, 1))\} \]
We let
\[ \mathcal{E} = \{ I^n, I^e, I^s, I^w, J^n, J^e, J^s, J^w \}; \]
thus \( \mathcal{E} \) is the set of open edges of the polygon \( \text{bdry} \ S \). Finally we let
\[ W, Z \]
be the closed triangles with vertices \((0, 0), (1, 0), (0, 1)\) and \((0, 0), (-1, 0), (0, -1)\), respectively; thus \( S \cup Z \cup W \) is the convex hull of \( S \).

6.1 Determination of \( \Gamma(S, r, s) \), \( 0 < r < \infty, \ 0 < s < \infty \).

6.2 The relevant arc geometry.

Definition 6.2.1. For \( 0 < s < \infty \) and \( \pi/4 \leq \theta \leq \pi/2 \) we let
\[ c_1(s, \theta) = s \sin \theta (1, 1), \quad a_1(s, \theta) = a(c_1(s, \theta), s, 5\pi/4 - (\theta - \pi/4), 5\pi/4 + (\theta - \pi/4)). \]
For \( 0 < r < 1, 0 < s < \infty \) and \( \pi/4 \leq \theta \leq \pi/2 \) we let
\[ c_2(r, s, \theta) = c_1(s, \theta) - (r + s)u(\theta), \quad a_2(r, s, \theta) = a(c_2(r, s, \theta), r, 0, \theta), \quad w(r, s, \theta) = s(\sin \theta - \cos \theta) + r(1 - \cos \theta) \]
and we note that
\[ w(r, s, \theta) = (c_2(r, s, \theta) + re_1) \cdot e_1. \]
For \(0 < r < \infty\) and \(0 < s < \infty\) we let
\[ s_3(r, s, \theta) = ((1, -r \sin \theta), (1, -1 + r)). \]
For \(0 < r < \infty\) we let
\[ a_4(r) = a((1 - r, -1 + r), r, -\pi/2, 0). \]
For \(0 < r < \infty\) and \(0 < s < \infty\) we let
\[ s_5(r) = ((s, 0), (1 - r, 0)). \]
For \(0 < r < \infty\) we let
\[ a_6(r) = a((1 - r, -r), r, 0, \pi/2). \]
For \(0 < r < \infty\) we let
\[ s_7(r) = ((1, -r), (1, -1 + r)). \]
For \(0 < r < \infty\) we let
\[ a_8(r) = ((r, -r), r, \pi, 3\pi/2). \]
For \(0 < r < \infty\) we let
\[ s_9(r) = ((r, 0), (1 - r, 0)). \]
We let
\[ W_1 = \{ (r, s) \in (0, \infty) \times (0, \infty) : (1 - \sqrt{2}/2)r < 1 < r + s \}, \]
\[ W_2 = \{ (r, s) \in (0, \infty) \times (0, \infty) : r + s \leq 1 \text{ and } 2r \leq 1 \}. \]

**Proposition 6.2.1.** There is one and only one function
\[ \Theta : W_1 \to (\pi/4, \pi/2) \]
such that whenever \(0 < r < \infty\), \(0 < s < \infty\) and \(\pi/4 < \theta < \pi/2\) we have
\[ w(r, s, \theta) = 1 \iff (r, s) \in W_1 \text{ and } \theta = \Theta(r, s). \]

**Proof.** Suppose \(0 < r < \infty\) and \(0 < r < \infty\). Then
\[ \lim_{\theta \to \pi/4} w(r, s, \theta) = r(1 - \sqrt{2}/2), \quad \lim_{\theta \to \pi/2} w(r, s, \pi/2) = r + s \]
and
\[ \frac{\partial}{\partial \theta} w(r, s, \theta) = s(\cos \theta + \sin \theta) + r \sin \theta > 0 \quad \text{whenever } \pi/4 < \theta < \pi/2. \]
\[ \Box \]
The following Proposition is evident.

**Proposition 6.2.2.** Suppose $0 < r < \infty$, $0 < s < \infty$ and $\pi/4 < \theta < \pi/2$. Then the circles containing $a_1(s, \theta)$ and $a_2(r, s, \theta)$ meet tangentially at

$$(s(\sin \theta - \cos \theta), 0).$$

Moreover, the circle containing $a_2(r, s, \theta)$ meets the line $l(e_1, 1)$ tangentially if and only $(r, s) \in W_1$ and $\theta = \Theta(r, s)$.

**Definition 6.2.2.** Let

$$W_3 = \{(r, s) \in W_1 : (1 + \sin \Theta(r, s))r \leq 1\}.$$

For each $(r, s) \in W_3$ we let $F_{r,s}$ be the compact subset of $\mathbb{R}^2$ whose boundary is the union of the closures of

$$\sigma[a_1(s, \Theta(r, s)) \cup a_2(r, s, \Theta(r, s)) \cup s_3(r, s, \Theta(r, s)) \cup a_4(r)], \quad \sigma \in \Sigma.$$

**Definition 6.2.3.** For each $(r, s) \in W_2$ we let $G_{r,s}$ be the compact subset of $\mathbb{R}^2$ whose boundary is the union of the closures of

$$\sigma[a_1(s, \pi/2) \cup s_5(r, s) \cup a_6(r) \cup s_7(r) \cup a_4(r)], \quad \sigma \in \Sigma.$$

We let $H_r$ be the convex hull of the union of the arcs

$$a((1-r, -r), r, 0, \pi/2), \quad a((1-r, 1+r), r, -\pi/2, 0), \quad a((r, -1+r), r, \pi, 3\pi/2), \quad a((r, -r), r, \pi, 3\pi/2).$$

We let

$$I_r = \sigma[H_r].$$

The following Theorem will be proved in Section ??.

**Theorem 6.2.1.** Suppose $0 < r < \infty$, $0 < s < \infty$,

$$E \in \Gamma(S, r, s), \quad E \neq \emptyset \quad \text{and} \quad \sigma[E] = E \quad \text{for} \quad \sigma \in \Sigma.$$

Then either $(r, s) \in W_3$ and

$$< E > = F_{r,s}$$

or $(r, s) \in W_2$ and

$$< E > = G_{r,s} \quad \text{or} \quad < E > = H_r.$$
6.2.1 Some lemmas.

Throughout this subsection we suppose

\[ 0 < r < \infty, \quad 0 < s < \infty, \quad E \in \Gamma(r, s, S) \sim \{0\}. \]

**Lemma 6.2.1.** \( E \cap (V \sim \{(0, 0)\}) = \emptyset. \)

*Proof.* This follows from \((\Gamma 0)\) and \((\Gamma 1)\). \( \square \)

**Lemma 6.2.2.** Suppose \( E \) is special. Then \( 0 \not\in \text{bdry} E \).

*Proof.* Suppose, contrary to the Lemma, \( 0 \in \text{bdry} E \). Let \( H = \text{Tan}(E, 0) \).

Then, as \( \sigma(0) = 0 \) for all \( \sigma \in \Sigma \), we have \( \sigma[H] = H \) for all \( \sigma \in \Sigma \) which is impossible since \( H \) is a closed halfspace and \( 0 \notin H \). \( \square \)

**Lemma 6.2.3.** Suppose \( A \in \text{int}(E, S) \), \( A \subset T \) and \( A \cap L = \emptyset \). Then there is \( \theta \in (\pi/4, \pi/2] \) such that either (i) \( A = a((1-r, -r \sin \theta), r, 0, \theta) \) or (ii) \( A = a(r \sin \theta, -1+r), r, 3\pi/2 - \theta, 3\pi/2) \).

*Proof.* We have either (iii) \( A \subset h((1, 1), e_1 + e_2) \) or (iv) \( A \subset h((1, 1), e_1 + e_2) \).

In case (iii) holds the circle containing \( A \) meets \( I^\infty \) tangentially at a point of \( \text{ends}(A) \) and (ii) holds and in case (iv) holds the circle containing \( A \) meets \( I^\infty \) tangentially at a point of \( \text{ends}(A) \) and (i) holds. \( \square \)

**Lemma 6.2.4.** Suppose \( A \in \text{cmp}(E, S) \). Then

\[ \text{ends}(A) \not\subseteq \text{cl} K \quad \text{for each} \quad K \in \mathcal{E}. \]

*Proof.* Suppose the Lemma were false. Replacing by \( E \) by \( \sigma[E] \) for some \( \sigma \in \Sigma \) we may assume that \( \text{ends}(A) \subset I^\infty \) or \( \text{ends}(A) \subset I^n \). Let \( C \) be the circle containing \( A \).

Suppose \( \text{ends}(A) \subset \text{cl} I^\infty \). Then \( A \in \text{int}(E, S) \). Since the length of \( A \) does not exceed \( \pi \) the circle \( C \) must meet the line \( \{x \in \mathbb{R}^2 : x_1 = 1\} \) transversely. This is incompatible with \((\Gamma 0)\) and \((\Gamma 1)\).

Suppose \( \text{ends}(A) \subset \text{cl} I^n \). Let \( a, b \) be such that \( \text{ends}(A) = \{a, b\} \) and \( d_1 < b_1 \). Note that \( b \in I^n \) since \( (1, 0) \notin E \). Let \( Y = L_1 \); let \( G = (\mathbb{R}^2 \sim \text{bdry} S) \cup I^n \), let \( X \) be the connected component of \( G \cap \text{bdry} E \) which contains \( b \) and let \( P, Q, R \), etc., be as in Theorem 2.2.1. Since \( C \) meets \( L_1 \) transversely we have \( b \in Q \); since \( Y \) is not a circle we infer that (II) of Theorem 2.2.1 holds. Let \( d \in Q \) be such that \( |d| \geq |q| \) whenever \( q \in Q \). Keeping in mind Theorem 2.2.1 (xiv) and (xv) there are \( B, B', d', d'' \) such that \( B, B' \in \text{cmp}(E, S) \), \( \text{ends}(B) = \{d', d\} \), \( \text{ends}(B') = \{d, d''\} \) and such that

\[ B \subset h(e_1, d_1) \quad \text{and} \quad B' \subset h(-e_1, d_1). \]  \( \tag{21} \)

and \( d'' \in \text{bdry} S \sim I^n \). If \( B \in \text{int}(E, S) \) then \( B' \in \text{ext}(E, S) \) and this forces \( d'' = (1, 0) \) which is impossible. If \( B \in \text{ext}(E, S) \) then \( B'' \in \text{int}(E, S) \) and \( d'' \in I^n \cup I^* \). If \( d'' \in I^* \) then \( D \) meets \( I^* \) tangentially which forces the length of \( B'' \) to exceed \( \pi r \). If \( d'' \in I^* \) then \( D \) meets \( I^* \) tangentially and this forces \( d_1 < 0 \) in view of 21. \( \square \)
Lemma 6.2.5. Suppose $E$ is special, $A \in \text{cmp}(E, S)$, $C$ is the circle containing $A$ and $c$ is the center of $C$. If $A \cap L^\pm \neq \emptyset$ then $c \in L^\pm$. 

Proof. Suppose $A \subset Z$ and let $e \in A \cap L^+$. Since $\sigma_\pm(e) = e$ we have either (i) $\sigma_\pm[\text{Tan(bdry } E, e), e] = L^-$ or (ii) $\sigma_\pm[\text{Tan(bdry } E, e), e] = L^+$. If (ii) held we would have either $A \subset h(u^-, e \bullet u^-)$ in which case (iii) $\text{ends}(A) \subset \text{cl} I^n$ or $A \subset h(-u^-, e \bullet u^-)$ in which case (iv) $\text{ends}(A) \subset \text{cl} J^2$. Neither (iii) nor (iv) is compatible with Lemma 6.2.4 so (i) holds and $c \in L^+$. 

Suppose $A \subset T$ and let $e \in A \cap L^-$. Since $\sigma_\pm(e) = e$ we have either (v) $\sigma_\pm[\text{Tan(bdry } E, e), e] = L^+$ or (vi) $\sigma_\pm[\text{Tan(bdry } E, e), e] = L^-$. If (vi) held we would have either (vii) $A \subset h(u^+, e \bullet u^+)$ or (vii) $A \subset h(-u^-, e \bullet u^-)$. If (vii) held there would be a $a \in \text{ends}(A) \cap I^e$ by Lemma 6.2.4. By $(\Gamma_0)$ and $(\Gamma_1)$ $C$ would meet $I^e$ tangentially at $a$ and that would force the length of $A$ to exceed $\pi r$. If (viii) held there would be $a \in \text{ends}(A) \cap I^e$ by Lemma 6.2.4. By $(\Gamma_0)$ and $(\Gamma_1)$ $C$ would meet $I^e$ tangentially at $a$ and that would force the length of $A$ to exceed $\pi r$. Thus (v) holds and $c \in L^-$. 

\[ \square \]

Lemma 6.2.6. Suppose $E$ is special. The following statements hold:

(I) If $A \in \text{int}(E, S)$, $A \subset T$ and $T \cap L^- \neq \emptyset$ then $A = A^{nw}$ or $A = A^{se}$.

(II) If $A \in \text{int}(E, S)$, $A \subset T$ and $A \cap L^- = \emptyset$ then for some $\theta \in (\pi/4, \pi/2]$ either $A = a_1(r, s, \theta)$ or $A = \sigma_\pm[a_1(r, s, \theta)]$.

(III) If $A \subset Z$ then $A = a_1(r, s, \theta)$ for some $\theta \in (\pi/4, \pi/2]$.

Proof. Suppose $A \in \text{cmp}(E, S)$, $C$ is the circle containing $A$ and $c$ is the center of $C$.

Suppose $A \subset T$ and $A \cap L^- \neq \emptyset$. By Lemma 6.2.5 we have $c \in L^-$. Since the length of $A$ does not exceed $\pi r$ there are $a, b$ such that $\text{ends}(A) = \{a, b\}$ and either (i) $a \in I^e$ and $b \in I^e$ or (ii) $a \in I^n$ and $b \in I^n$. If (i) holds then by $(\Gamma_0)$ and $(\Gamma_1)$ the circle $C$ meets $I^e$ tangentially at $a$. Since the length of $A$ does not exceed $\pi r$ we conclude that $A = A^{se}$. By a similar argument we find that if $C$ meets $I^n$ tangentially at $a$ then $A = A^{nw}$.

So suppose $C$ meets $I^n$ transversely at $a$. Then there is $A' \in \text{ext}(E, S)$ such that $a \in \text{ends}(A')$. Let $C'$ be the circle containing $A'$ and let $c'$ be its center. By Lemma 6.2.4 we have $A' \cap L^+ \neq \emptyset$. By Lemma 6.2.4 $c' \in L^+$. Since $c, a, c'$ are collinear we infer that either the length of $A$ exceeds $\pi r$ or the length of $A'$ exceeds $\pi s$. Thus (I) holds.

Suppose $A \subset T$ and $A \cap L^- = \emptyset$. The either (iii) $A \subset h(-u^-, e \bullet u^-)$ or (iv) $A \subset h(u^-, e \bullet u^-)$. Suppose (iii) holds. By Lemma 6.2.4 there are $a, b$ such that $\text{ends}(A) = \{a, b\}$, $a \in \text{cl}(I^p)$ and $b \in I^e$. Since $C$ meets $I^e$ tangentially at $b$ and we find that there is $\theta \in (\pi/4, \pi/2]$ such that $A = a_2(r, s, \theta)$. In case (iv) holds we replace $E$ by $\sigma_\pm[E]$ and deduce that $\sigma_\pm[A] = a_2(r, s, \theta)$ for some $\theta \in (\pi/4, \pi/2]$. 

Suppose $A \in \text{ext}(E, S)$ and $A \subset Z$. By Lemma 6.2.4 there is $A \cap L^+ \neq \emptyset$ so $c \in L^+$. In particular, if $a = e - se_2$ and $b = e - se_1$ then $\text{ends}(A) = \{a, b\}$. Since $V \cap \text{bdry } S = \emptyset$ we have $a \in I^n$ and $b \in J^e$. If $C$ meets $L_1$ tangentially
at a then \( A = a_1(s, \pi/2) \) since the length of \( A \) does not exceed \( \pi s \). So suppose \( C \) meets \( I^n \) transversely at \( a \). Then there is \( A' \in \text{int}(E, S) \) such that \( a \in \text{ends}(A') \). It follows from (I) and (II) that there is \( \theta \in (\pi/4, \pi/2] \) such that \( A' = a_2(r, s, \theta) \). Given that \( c \in L^+ \) we may now conclude with the help of (Γ5) that \( A = a_1(s, \theta) \).

\[ \Box \]

**Lemma 6.2.7.** Suppose \( E \) is special, \( A \in \text{ext}(E, S) \) and \( A \subset Z \). Then

\[ (E \sim S) \cap Z = \{tx : 0 < |t| \leq 1 \text{ and } x \in A\}. \]

**Proof.** Replacing \( E \) by \( \sigma_\pm[E] \) if necessary we may assume that \( A \subset Z \). By 6.2.6 there is \( \theta \in (\pi/4, \pi/2] \) such that \( A = a_1(s, \theta) \). Keeping in mind (Γ5), we find that if the Lemma were false there would be \( x \in A \) and \( 0 < t_1 < t_2 < 1 \) such that \( (t_1 x, t_2 x) \cap E = \emptyset \) and \( [t_1 x, x] \subset E \). Let \( A' \in \text{ext}(E, S) \) be such that \( t_2 x \in A' \). Applying Lemma 6.2.6 we obtain \( \theta' \in (\pi/4, \pi/2] \) such that \( A' = a_1(s, \theta') \). Since \( (t_1 x, t_2 x) \cap E = \emptyset \) we find that (Γ5) does not with \( A \) there replaced by \( A' \). \[ \Box \]

**Theorem 6.2.2.** Suppose \( E \) is special and \( 0 \not\in E \). Then

\[ E \in \{0, H_s, J_s, I_s\}. \]

**Proof.** From Lemma 6.2.7 we find that \( E \subset S \). It follows that \( E \cap T \in \Gamma(T, r, s) \) and \( E \cap U \subset \Gamma(U, r, s) \). We leave the remaining details of the proof to the reader. \[ \Box \]

**Theorem 6.2.3.** Suppose \( E \) is special and \( 0 \in E \). Then either \( (r, s) \in W_3 \) and \( E = F_{r,s} \) or \( (r, s) \in W_2 \) and \( E = G_{r,s} \).

**Proof.** Suppose \( E \neq \emptyset \). Since \( E \cap \text{int} Z \neq \emptyset \) and \( E \cap \text{int} W \neq \emptyset \) we infer from Lemma 6.2.6 that for each \( i = 1, 2 \) there is \( \theta_i \in (\pi/4, \pi/2] \) such that \( A_i = a_1(r, s, \theta_i) \in \text{ext}(E, S) \). From Lemma 6.2.7 we infer that

\[ (E \sim S) \cap Z = \{tx : 0 < |t| \leq 1 \text{ and } x \in A_1\} \text{ and } (E \sim S) \cap W = \{tx : 0 < |t| \leq 1 \text{ and } x \in A_2\} \]

Let \( a^1 = (s(\sin \theta - \cos \theta), 0) \in \text{ends}(A_1) \) and let \( a^2 = (0, s(\sin \theta - \cos \theta)) \in \text{ends}(A_2) \).

**Case One.** \( \theta_1 = \pi/2 \). Let \( Y = L_1 \), let \( G = (\mathbb{R}^2 \sim \text{bdry} S) \cup I^n \) and let \( X \) be the connected component of \( G \) which contains \( a \). We infer from (22), Lemma 6.2.6 and Theorem 2.2.1 that \( r + s \leq 1 \) and that

\[ X = A \cup [(s, 0), (1-r, 0)] \cup A^\text{ne}. \]

Let \( Y = h(e_1, 1) \), let \( G = (\mathbb{R}^2 \sim \text{bdry} S) \cup I^n \) and let \( X \) be the connected component of \( G \) which contains \( (1, -r) \). Applying Theorem 2.2.1 we infer that \( 2r \leq 1 \) so \( (r, s) \in W_2 \) and

\[ X = A^\text{ne} \cup [(1-r), (1, 1+r)] \cup A^\text{ne}. \]

Let \( Y = L_2 \), let \( G = (\mathbb{R}^2 \sim \text{bdry} S) \cup I^w \) and let \( X \) be the connected component of \( G \) which contains \( a^2 \). Keeping in mind (22) we apply Theorem
2.2.1 to obtain $B_2 \in \text{int}(E, S)$ and $b^2 \in I^w \cap \text{ends}(B_2)$ such that $B_2 \subseteq h(e_2, b^2 \cdot e_2)$ and the circle containing $B_2$ meets $I^w$ tangentially at $b^2$. Since $r + s \leq 1$ we infer from Lemma 6.2.6 (II) that $\theta_2 = \pi/2$ and $B_2 = A^w$. Arguing as in the preceding paragraph we infer that $\text{bdry} \ G_{r,s} \subseteq \text{bdry} \ E$. Keeping in mind (22) and Lemma 6.2.6 we infer that $\text{bdry} \ G_{r,s} \subseteq \text{bdry} \ E$.

**Case Two.** $\theta_1 < \pi/2$. In this case there is $B_1 \in \text{int}(E, S)$ such that $a^1 \in \text{ends}(A_1) \cap \text{ends}(B_1)$. From Lemma 6.2.6 (II) we infer that $B_1 = a_2(r, s, \theta_1)$ and that $r + s > 1$. Let $Y = h(e_1, 1)$, let $G = (\mathbb{R}^2 \sim \text{bdry} \ S) \cup I^w$ and let $X$ be the connected component of $G$ which contains $(1, -r \sin \theta_1)$. By Theorem 2.2.1 and Lemma 6.2.6 (II) we infer that

$$X = B_1 \cup \left[ b^1, (1, -r \sin \theta_1) \right] \cup A^w,$$

and $(1 + r \sin \theta_1) \leq 1$ so $(r, s) \in W_3$. Were it the case that $\theta_2 = \pi/2$ we could let $Y = L_2$, let $G = (\mathbb{R}^2 \sim \text{bdry} \ S) \cup I^w$ and let $X$ be the connected component of $G$ which contains $a^2$ and apply Theorem 2.2.1 with (22) in mind to infer that $r + s \leq 1$ which we have excluded. Thus $\theta_2 < \pi/2$ and we may proceed as in the preceding paragraph to infer that $\text{bdry} \ F_{r,s} \subseteq \text{bdry} \ S$. Given the (22) and the constraints on the members of $\text{int}(E, S)$ we infer that $\text{bdry} \ F_{r,s} \subseteq \text{bdry} \ S$. □

### 6.2.2 Lengths and areas for $F_{r,s}$.

Suppose $(r, s) \in W_3$ and let $\theta = \Theta(r, s)$. Let

$$L = H^4(\text{bdry} \ F_{r,s}), \quad A_{\text{out}} = L^2(F_{r,s} \sim S), \quad A_{\text{in}} = L^2(F_{r,s} \cap S).$$

Let $L_1, L_2, L_3, L_4$ be the lengths of $a_1(s, \theta), a_2(r, s, \theta), s_3(r, s), a_4(r)$, respectively. Then

$$L_1 = \left( \frac{2\theta - \pi}{2} \right) s, \quad L_2 = \theta r, \quad L_3 = 1 - r(1 + \sin \theta), \quad L_4 = \frac{\pi}{2} r$$

so

$$L = 2L_1 + 4L_2 + 4L_3 + 2L_4 = \pi(r - s) + 4(\pi/2)\theta.)$$

Let $A_1, A_2, A_3$ be the areas of the square with diagonal $[(0, 0), c_1(s, \theta)]$; the rectangle with diagonal $[(s(\sin \theta - \cos \theta), 0), c_1(s, \theta)]$; and the sector $\{tx : 0 < t < 1\}$ and $x \in a_1(s, \theta)$, respectively. Then

$$A_1 = (s \sin \theta)^2, \quad A_2 = (s \cos \theta)(s \sin \theta), \quad A_3 = \left( \frac{\pi}{2} - 2\theta \right) \frac{s^2}{2}$$

so

$$A_{\text{out}} = 2(A_1 - (A_2 + A_3)) = s^2 \left( 2\sin \theta(\sin \theta - \cos \theta) - \left( \frac{\pi}{2} - 2\theta \right) \right).$$

Let $A_4, A_5, A_6, A_7, A_8$ be the areas of the rectangle with diagonal $[c_2(r, \theta), (1, 0)]$; the triangle with vertices $c_2(r, s, \theta), (c_2(r, s, \theta) \cdot e_1, 0), (s \sin \theta, 0)$; the sector $\{tx :
$0 < t < 1$ and $x \in a_2(r, s, \theta)$; the square with diagonal $[(1 - r, -1 + r), (1, -1)]$; and the sector $\{ tx : 0 < t < 1$ and $x \in a_4(r) \}$, respectively. Then

$$A_4 = r(r \sin \theta), \quad A_5 = \frac{1}{2} (r \cos \theta)(r \sin \theta), \quad A_6 = \frac{\theta r^2}{2}, \quad A_7 = r^2, \quad A_8 = \frac{\pi r^2}{2}$$

so

$$A_{\text{in}} = 2 - 4(A_4 - (A_5 + A_6)) - 2(A_7 - A_8) = 2 + r^2(\sin \theta(2 \cos \theta - 4) + 2\theta - 2 + \frac{\pi}{2}).$$

### 6.2.3 Lengths and areas for $G_{r, s}$.

Suppose $(r, s) \in W_2$. Let

$$L = \mathcal{H}^1(\text{bdry } G_{r, s}), \quad A_{\text{out}} = \mathcal{L}^2(G_{r, s} \sim S), \quad A_{\text{in}} = \mathcal{L}^2(G_{r, s} \cap S).$$

Let $L_1, L_2, L_3, L_4, L_5$ be the lengths of $a_1(s, \pi/2)$, $s_5(r, s)$, $a_6(r)$, $s_7(r)$, $a_4(r)$, respectively. Then

$$L_1 = \frac{\pi}{2} r, \quad L_2 = 1 - (r + s), \quad L_3 = \frac{\pi}{2} r, \quad L_4 = 1 - 2r, \quad L_5 = \frac{\pi}{2} r$$

so

$$L = 2L_1 + 4L_2 + 4L_3 + 4L_4 + 2L_5 = 2\pi + 8 + r(3\pi - 12) + 4s.$$

Also,

$$A_{\text{out}} = 2s^2 \left( 1 - \frac{\pi}{4} \right) \quad \text{and} \quad A_{\text{in}} = 2 - 6r^2 \left( 1 - \frac{\pi}{4} \right).$$

### 6.2.4 Lengths and areas for $H_{r, s}$.

Suppose $(r, s) \in W_2$. Let

$$L = \mathcal{H}^1(\text{bdry } H_r), \quad A_{\text{out}} = \mathcal{L}^2(H_r \sim S), \quad A_{\text{in}} = \mathcal{L}^2(H_r \cap S).$$

Let $L_1, L_2, L_3, L_4, L_5$ be the lengths of $a_8(r, \pi/2)$, $s_9(r)$, $a_6(r)$, $s_7(r)$, $a_4(r)$, respectively. Then

$$L_1 = \frac{\pi}{2} r, \quad L_2 = 1 - 2r, \quad L_3 = \frac{\pi}{2} r, \quad L_4 = 1 - 2r, \quad L_5 = \frac{\pi}{2} r$$

so

$$L = 2L_1 + 4L_2 + 4L_3 + 4L_4 + 2L_5 = 8 + r(4\pi - 16).$$

Also,

$$A_{\text{out}} = 0 \quad \text{and} \quad A_{\text{in}} = 2 - 8r^2 \left( 1 - \frac{\pi}{4} \right).$$
6.3 The Chan-Esedoglu functional.

Suppose $0 < \epsilon < \infty$.

Let $\epsilon_1 = \frac{2 - \sqrt{\pi}}{4 - \pi}$.

**Theorem 6.3.1.** We have $D \in n_{\epsilon}^{loc}(V_S)$ and $D = \langle D \rangle$ if $
\epsilon < \epsilon_1$; $D \in \{\emptyset, G_{\epsilon,\epsilon}, H_{\epsilon}\}$ if $\epsilon = \epsilon_1$; $D \in \emptyset$ if $\epsilon > \epsilon_1$.

**Proof.** Suppose $D \in n_{\epsilon}^{loc}(V_S)$ and $D = \langle D \rangle$. Then $D$ is special by Theorem 2.1.1. From ?? and ?? we find that $D \in \{\emptyset, F_{\epsilon,\epsilon}\}$ if $(\epsilon, \epsilon) \in W_3$, $D \in \{\emptyset, G_{\epsilon,\epsilon}, H_{\epsilon}, I_{\epsilon}, J_{\epsilon}\}$ if $(\epsilon, \epsilon) \in W_2$, $D \in \emptyset$ if $(\epsilon, \epsilon) \notin W_3 \cup W_2$.

Suppose $(\epsilon, \epsilon) \in W_3$ and let $\theta = \Theta(\epsilon, \epsilon)$. Let $p(\theta) = 8\theta - \pi - \cos \theta(2\cos \theta + 4\sin \theta)$ for $\theta \in \mathbb{R}$.

Since $p > 0$ on $(\pi/4, \pi/2)$ and since $2\epsilon > 1/2$ we find that

$$(V_S)_{\epsilon}(F_{\epsilon,\epsilon})$$

$$= \epsilon L(\epsilon, \epsilon) - A_{in}(\epsilon, \epsilon) + A_{out}(\epsilon, \epsilon)$$

$$= p(\theta)\epsilon^2 + 4\epsilon - 2$$

$$\geq p(\theta)\frac{\epsilon^2}{2} + 4\epsilon - 2$$

$$> 0$$

$$= (V_S)_{\epsilon}(\emptyset)$$

so $F_{\epsilon,\epsilon} \notin n_{\epsilon}^{loc}(V_S)$.

Suppose $(\epsilon, \epsilon) \in W_2$. We have

$$(V_S)_{\epsilon}(I_{\epsilon})$$

$$= (V_S)_{\epsilon}(F_{\epsilon,\epsilon})$$

$$= 2(4(1 - 2\epsilon) + 2\pi\epsilon) - 2(1 - 4(\epsilon^2 - \pi/4\epsilon^2))$$

$$= 2((\pi - 4)\epsilon^2 + 4\epsilon - 1)$$

$$\begin{cases} < 0 & \text{if } \epsilon < \epsilon_1, \\
= 0 & \text{if } \epsilon = \epsilon_1, \\
> 0 & \text{if } \epsilon > \epsilon_1. \end{cases}$$
6.4 The ROF functional.

Suppose $0 < \epsilon < \infty$.

Let

$$F(g) = \frac{1}{2} \int_{\mathbb{R}^2} |g - 1_S|^2 \, d\mathcal{L}^2 \quad \text{for } g \in \mathcal{F}(\mathbb{R}^2).$$

Let $U_y, 0 < y < \infty$ be as in Section 1.2; thus

$$U_y(D) = y\mathcal{L}^2(S \sim E) - (1 - y)\mathcal{L}^2(E \cap S) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).$$

Suppose $f \in m_e(F)$ and let

$$D_y = \langle \{ f > y \} \rangle \quad \text{for } 0 < y < \infty.$$ 

From ?? we have $D_y = \emptyset$ if $1 < y < \infty$.

We will determine $D_y, 0 < y < 1$.

Let $r(y) = \frac{\epsilon}{1 - y}$ and $s(y) = \frac{\epsilon}{y}$ for $0 < y < 1$.

Evidently,

$$\frac{1}{\epsilon} = \frac{1}{r(y)} + \frac{1}{s(y)} \quad \text{whenever } 0 < y < 1.$$ 

From ?? and ?? we have

$$D_y \begin{cases} \in \Gamma(S, r(y), s(y)) & \text{if } 0 < y < 1, \\ = \emptyset & \text{if } 1 < y < \infty. \end{cases}$$

For each $i = 1, 2, 3$ let

$$Y_i = \{ y \in (0, 1) : (r(y), s(y)) \in W_i \}.$$ 

Let

$$\vartheta(y) = \Theta(r(y), s(y)) \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \quad \text{for } y \in Y_1.$$ 

Since $F$ and therefore $F_e$ is strictly convex we have

$$\int_{\mathbb{R}^2} |f \circ \sigma - f| \quad \text{for } \sigma \in \Sigma$$

which implies that whenever $0 < y < \infty$ we have

$$\sigma[E_y] = E_y \quad \text{for } \sigma \in \Sigma \quad \text{and } E_y \text{ is special;}$$

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it follows that whenever $0 < y < 1$ we have

$$E_y \begin{cases} 
\in \{\emptyset, F_{r(y), s(y)}\} & \text{if } (r(y), s(y)) \in Y_3; \\
\in \{\emptyset, G_{r(y), s(y)}\} & \text{if } (r(y), s(y)) \in Y_2; \\
= \emptyset & \text{if } (r(y), s(y)) \not\in Y_2 \cup Y_3.
\end{cases}$$

We let

$$\epsilon_1 = \frac{1}{4}, \quad \epsilon_2 = \left(1 + \frac{\sqrt{2}}{2}\right)^{-1}, \quad \epsilon_3 = \left(1 - \frac{\sqrt{2}}{2}\right)^{-1}.$$

Evidently,

$$0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \infty.$$

We let

$$Q(\theta, t) = t^2 + b(\theta)t + c(\theta) \quad \text{for } (\theta, t) \in \mathbb{R}^2$$

where we have set

$$b = \sin\left(-\frac{1}{\epsilon} + 1\right), \quad \text{we let } \quad c = \frac{1}{\epsilon}(1 - \cos).$$

**Proposition 6.4.1.** We have

$$Y_1 = \begin{cases} 
\emptyset & \text{if } \epsilon \geq \epsilon_3; \\
(0, 1 - \frac{\sqrt{1 - 4\epsilon}}{\epsilon_3}) & \text{if } \epsilon_1 < \epsilon < \epsilon_3; \\
\left(\frac{1}{2}, 1 - \frac{\sqrt{1 - 4\epsilon}}{\epsilon_3}\right) & \text{if } \epsilon = \epsilon_1; \\
(0, 1 - \frac{\sqrt{1 - 4\epsilon}}{\epsilon_3}) \cup \left(\frac{1 + \sqrt{1 - 4\epsilon}}{2}, 1 - \frac{\sqrt{1 - 4\epsilon}}{\epsilon_3}\right) & \text{if } 0 < \epsilon < \epsilon_1
\end{cases}$$

and

$$Y_2 = \begin{cases} 
\emptyset & \text{if } \epsilon > \epsilon_1; \\
\left\{\frac{1}{2}\right\} & \text{if } \epsilon = \epsilon_1; \\
\left[\frac{1 + \sqrt{1 - 4\epsilon}}{2}, 1 - 2\epsilon\right] & \text{if } \epsilon < \epsilon_1.
\end{cases}$$

Moreover,

$$Q\left(\vartheta(y), \frac{1}{r(y)}\right) = 0 \quad \text{for } y \in Y_1.$$

**Proof.** Suppose $0 < y < 1$. Then $(1 - \sqrt{2}/2)r(y) < 1$ if and only if $y < 1 - \epsilon/\epsilon_3$ and $2r(y) \leq 1$ if and only if $y \geq 1 - 2\epsilon$. Furthermore,

$$1 < r(y) + s(y) \iff 0 < y^2 - y + \epsilon.$$

Since

$$\{y \in \mathbb{R} : 0 \leq y^2 - y + \epsilon\} = \begin{cases} 
\left[\frac{1 - \sqrt{1 - 4\epsilon}}{2}, \frac{1 + \sqrt{1 - 4\epsilon}}{2}\right] & \text{if } \epsilon < \epsilon_1, \\
\left\{\frac{1}{2}\right\} & \text{if } \epsilon = \epsilon_1, \\
\mathbb{R} & \text{if } \epsilon > \epsilon_1,
\end{cases}$$

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the formulae for $Y_1$ and $Y_2$ follow.

Suppose $y \in Y_1$. Then with $r = r(y)$, $s = s(y)$, $\theta = \vartheta(y)$ we have

\[
\frac{w(r, s, \theta)}{rs} = \frac{1}{r} (\sin \theta - \cos \theta) + \left(\frac{1}{\epsilon} - \frac{1}{r}\right) (1 - \cos \theta) - \frac{1}{r} \left(\frac{1}{\epsilon} - \frac{1}{r}\right) = Q \left(\theta, \frac{1}{r}\right).
\]

\[
6.4.1 \text{ Some preliminary calculations.}
\]

We let

\[
\Delta = b^2 - 4c
\]

and we let

\[
I = \left\{ \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] : \Delta(\theta) \geq 0 \right\}.
\]

Note that

\[
(b, c) \left(\frac{\pi}{4}\right) = \left(\frac{1}{\epsilon}, \frac{1}{\epsilon^3}\right) \quad \text{and} \quad (b, c) \left(\frac{\pi}{2}\right) = \left(- \frac{1}{\epsilon}, \frac{1}{\epsilon}\right).
\]

We have

\[
\Delta \left(\frac{\pi}{4}\right) = \left(\frac{1}{\epsilon} - \frac{1}{\epsilon^3}\right)^2 \quad \text{and} \quad \Delta \left(\frac{\pi}{2}\right) = \frac{1 - 4\epsilon}{\epsilon^2}.
\]

Now $\Delta' = 2b \cos(4/\epsilon) \sin \theta$ so $\Delta$ is decreasing on $(\pi/4, \pi/2)$. Thus $I$ is an interval and $\Delta$ is positive on the interior of $I$. Let

\[
\Upsilon = \sup I;
\]

evidently,

\[
\Upsilon = \frac{\pi}{2} \quad \text{if} \quad \epsilon \leq \epsilon_1 \quad \text{and} \quad \Upsilon < \frac{\pi}{2} \quad \text{if} \quad \epsilon > \epsilon_1.
\]

Let

\[
t_{\pm} = \frac{-b \pm \sqrt{\Delta}}{2} : \left[\frac{\pi}{4}, \Upsilon\right] \to \mathbb{R}.
\]

Since $b' > 0$ and $c' > 0$ on $(\pi/4, \pi/2)$ it follows from 1.6 that

\[
t'_{\pm} < 0 \quad \text{and} \quad t'_{-} > 0 \quad \text{on} \quad (\frac{\pi}{4}, \Upsilon).
\]

Thus

\[
rng (t_{\pm}) = \begin{cases} \left(\frac{1 + \sqrt{1 - 4\epsilon}}{2\epsilon}, \frac{1}{\epsilon}\right) & \text{if} \quad \epsilon \leq \epsilon_1, \\ \left(\frac{1}{2} \left(\frac{1}{\epsilon} + 1 - \sin \Upsilon\right), \frac{1}{\epsilon^3}\right) & \text{if} \quad \epsilon_1 < \epsilon. \end{cases}
\]

We have

\[
\left(\frac{t_{\pm}}{\sin}\right)^2 + \frac{b}{\sin} \left(\frac{t_{\pm}}{\sin}\right) + \frac{c}{\sin^2} = 0.
\]

Since

\[
\left(\frac{b}{\sin}\right)' = \frac{1 + \epsilon \cos}{\epsilon \sin^2} > 0 \quad \text{and} \quad \left(\frac{c}{\sin^2}\right)' = \frac{1}{\epsilon} \frac{\sin}{(1 + \cos)^2}.
\]

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we infer from 1.6 that
\[
\left( \frac{t_+}{\sin} \right)' < 0 \quad \text{and} \quad \left( \frac{t_-}{\sin} \right)' > 0 \quad \text{on} \ ( \frac{\pi}{4}, \Upsilon ).
\]

It follows that \((1 + \sin)/t_-\) is decreasing so that
\[
\frac{1 + \sin}{t_-} \geq \frac{\epsilon_3}{\epsilon_2} > 1 \quad \text{on} \ I.
\]

Also, \((1 + \sin)/t_+\) is increasing so
\[
\text{rng} \ \frac{1 + \sin}{t_+} = \begin{cases} 
\left( \frac{\epsilon}{\epsilon_2}, \frac{4\epsilon}{1 + \sqrt{1 - 4\epsilon}} \right) & \text{if} \ \epsilon \leq \epsilon_1, \\
\left( \frac{\epsilon}{\epsilon_2}, \frac{2 \{1 + \sin \Upsilon \}}{(\frac{1}{\epsilon} + 1) - \sin \Upsilon} \right) & \text{if} \ \epsilon_1 < \epsilon.
\end{cases}
\]

Let
\[J = \left\{ \theta \in I : \frac{1 + \sin \theta}{t_+(\theta)} \leq 1 \right\},\]

note that \(J\) is an interval and let \(v = \sup J\).

If \(\epsilon \leq \epsilon_1\) then
\[
\frac{1 + \sin}{t_+} \leq \frac{4\epsilon}{1 + \sqrt{1 - 4\epsilon}} \leq 1
\]
so, in this case, \(v = \pi/2\).

If \(\epsilon_1 < \epsilon\) we have
\[
2 \frac{1 + \sin \Upsilon}{(\frac{1}{\epsilon} + 1) - \sin \Upsilon} = 2 \frac{1 + \sin \Upsilon}{\sqrt{4(1 - \cos \Upsilon)}/\epsilon} = \sqrt{\epsilon} \frac{1 + \sin \Upsilon}{\sqrt{1 - \cos \Upsilon}} \geq 2\sqrt{\epsilon} > 1.
\]

It follows that \(\pi/4 < v < \pi/2\).

Finally, if \(\epsilon_2 \leq \epsilon\) we have
\[
\frac{1 + \sin \theta}{t_+(\theta)} > 1
\]
so, in this case, \(J = \emptyset\). Thus
\[
v = \frac{\pi}{2} \quad \text{if} \ \epsilon \leq \epsilon_1;
\]
\[
\frac{\pi}{2} < v < \frac{\pi}{2} \quad \text{if} \ \epsilon_1 < \epsilon < \epsilon_2;
\]
\[
J = \emptyset \quad \text{if} \ \epsilon_2 \leq \epsilon.
\]

**Proposition 6.4.2.** Let
\[
u_\pm = \frac{1}{\epsilon} - t_\pm.
\]

Then \(u_+ > 0, \ u_- > 0, \)
\[
\left( \frac{\sin - \cos}{u_+} \right)' < 0 \quad \text{and} \quad \left( \frac{\sin - \cos}{u_-} \right)' > 0 \quad \text{on} \ ( \frac{\pi}{4}, \Upsilon ).
\]
Proof. We have

\[ u_+ u_- = \frac{1}{\epsilon^2} - \frac{t_+ + t_-}{\epsilon} + t_+ t_- = \frac{1}{\epsilon^2} + \frac{b}{\epsilon} + c = \frac{\sin - \cos}{\epsilon} \]

so

\[ \left( \frac{\sin - \cos}{u_\pm} \right)' = (\epsilon u_\mp)' = -\epsilon t_\mp. \]

6.4.2

Proposition 6.4.3. We have

\[ Y_3 = \begin{cases} \emptyset & \text{if } \epsilon \geq \epsilon_2, \\ \{ y \in Y_1 : \vartheta(y) < v \} & \text{if } \epsilon < \epsilon_2. \end{cases} \]

Moreover, if \( \epsilon < \epsilon_2 \) and \( y \in Y_3 \) then

\[ (r(y), s(y)) = \left( \frac{1}{t_+(\vartheta(y))}, \frac{1}{u_+(\vartheta(y))} \right). \]

Proof. This follows directly from the foregoing.

Proposition 6.4.4. Suppose \( 0 < y_1 < y_2 < 1 \). If either (i) \( y_1, y_2 \in Y_3 \), or \( y_1, y_2 \in Y_2 \) or \( y_1 \in Y_3 \) and \( y_2 \in Y_2 \) then

\[ F_{r(y_2),s(y_2)} \subset F_{r(y_1),s(y_1)}. \] (23)

Proof. For \( y \in Y_3 \) let \( a(y) = (\sin \vartheta(y) - \cos \vartheta(y), 0) \) and let \( b(y) = (1, r(y) \sin \vartheta(y)) \). From ?? and ?? we have that \( a(y) \) is decreasing, \( b(y) \) is increasing, \( r(y) \) is increasing and \( s(y) \) is decreasing as \( y \) increases. This implies (23) if (i) holds. That (23) holds if (ii) holds is obvious. Finally, if \( Y_2 \neq \emptyset \), (i) implies that \( \operatorname{cl} \cup_{y \in Y_3} F_{r(y),s(y)} = G_{r(y),s(y)} \) so (iii) now follows from (ii).

For \( y \in (0, \infty) \) let

\[ u(y) = \begin{cases} (U_y)_r (F_{r(y),s(y)}) & \text{if } y \in Y_3, \\ (U_y)_r (G_{r(y),s(y)}) & \text{if } y \in Y_2, \\ 0 & \text{if } y \in (0, \infty) \sim (Y_2 \cup Y_3). \end{cases} \]

Let

\[ G_3 = \{(x, y) \in \mathbb{R}^2 \times Y_3 : x \in F_{r(y),s(y)} \text{ and } u(y) < 0\}, \]
\[ G_2 = \{(x, y) \in \mathbb{R}^2 \times Y_2 : x \in G_{r(y),s(y)} \text{ and } u(y) < 0\}, \]
\[ G = G_2 \cup G_3. \]

For each \( x \in \mathbb{R}^2 \) let

\[ g(x) = \sup \{ y \in Y_3 : x \in F_{r(y),s(y)} \} \cup \{ y \in Y_2 : x \in G_{r(y),s(y)} \}. \]

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Then, in view of the preceding Proposition,

\[ \{ y > y \} = \begin{cases} F_{r(y),s(y)} & \text{if } y \in Y_3, \\ G_{r(y),s(y)} & \text{if } y \in Y_2, \\ \emptyset & \text{if } y \notin Y_3 \cup Y_2. \end{cases} \]

**6.4.3**

Let

\[ \Phi(y) = U_y(F_{r(y),s(y)}) \quad \text{for } y \in Y_3. \]

Then

\[ \Phi(y) = \epsilon L(r(y), s(y)) + (y - 1)A_{in}(r(y), s(y)) + yA_{out}(r(y), s(y)) \]
\[ = \frac{\epsilon}{2r(y)(\epsilon - r(y))} \left( c_3(\theta)r(y)^3 + c_2(\theta)r(y)^2 + c_1(\theta)r(y) + c_0(\theta) \right) \]

where for \( \theta \in \mathbb{R} \) we have set

\[ c_3(\theta) = 4 \cos \theta \sin \theta + 4 - 4\theta - \pi, \]
\[ c_2(\theta) = \epsilon(-8\theta + 4\pi - 8 + 4\cos^2 \theta) - 8, \]
\[ c_1(\theta) = 4 + 8\epsilon, \]
\[ c_0(\theta) = -4\epsilon. \]

**6.4.4**

Let

\[ \Psi(y) = U_y(G_{r(y),s(y)}) \quad \text{for } y \in Y_2. \]

Then

\[ \Psi(y) = \frac{c_3y^3 + c_2y^2 + c_1y + c_0}{2y(1 - y)} \]

where

\[ c_3 = -4, \]
\[ c_2 = 8 - 16\epsilon, \]
\[ c_1 = (2\pi - 8)e^2 + 16\epsilon - 4, \]
\[ c_0 = (\pi - 4)e^2. \]

**References**


[AW1] W. K. Allard: *Whatever*

