1

# TOTAL VARIATION REGULARIZATION FOR IMAGE DENOISING; III. EXAMPLES.

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Abstract. Let

$$\mathcal{F}(\mathbb{R}^2) = \{f \in \mathbf{L}_\infty(\mathbb{R}^2) \cap \mathbf{L}_1(\mathbb{R}^2) : f \geq 0\}.$$

Suppose $s \in \mathcal{F}(\mathbb{R}^2)$ and $\gamma : \mathbb{R} \to [0, \infty)$. Suppose $\gamma$ is zero at zero, positive away from zero and convex. For $f \in \mathcal{F}(\Omega)$ let

$$F(f) = \int_\Omega \gamma(f(x) - s(x))\, d\mathcal{L}^2 x;$$

$\mathcal{L}^2$ here is Lebesgue measure on $\mathbb{R}^2$. In the denoising literature $F$ would be called a *fidelity* in that it measures how much $f$ differs from $s$ which could be a noisy grayscale image. Suppose $0 < \epsilon < \infty$ and let

$$\mathbf{n}_\epsilon^{loc}(F)$$

be the set of those $f \in \mathcal{F}(\mathbb{R}^2)$ such that $\mathbf{TV}(f) < \infty$ and

$$\epsilon \mathbf{TV}(f) + F(f) \leq \epsilon \mathbf{TV}(g) + F(g) \quad \text{for } g \in \mathbf{k}(f);$$

here $\mathbf{TV}(f)$ is the total variation of $f$ and $\mathbf{k}(f)$ is the set of $g \in \mathcal{F}(\mathbb{R}^2)$ such that $g = f$ off some compact subset of $\mathbb{R}^2$. A member of $\mathbf{m}_\epsilon^{loc}(F)$ is called a *total variation regularization of $s$ (with smoothing parameter $\epsilon$)*. Rudin, Osher and Fatemi in [ROF] and Chan and Esedoglu in [CE] have studied total variation regularizations of $F$ where $\gamma(y) = y^2$ and $\gamma(y) = y$, $y \in \mathbb{R}$, respectively.

*Our purpose in this paper is to describe, in complete detail, $\mathbf{m}_\epsilon^{loc}(F)$ when $s$ is the indicator function of* either

(1) $$S = ([0,1] \times [0,-1]) \cup ([-1,0] \times [0,1])$$

or

(2) $$S = \{x \in \mathbb{R}^2 : |x - \mathbf{c}_+| \leq 1\} \cup \{x \in \mathbb{R}^2 : |x - \mathbf{c}_-| \leq 1\}$$

where, for some $l \in [1, \infty)$, $\mathbf{c}_\pm = (\pm l, 0)$.

We believe these examples reveal a great deal about the nature of total variation regularizations. In addition, one can test computational schemes for total variation regularization against these examples.

## Contents

1. Introduction. 2
1.1. Total variation. 2
1.2. Total variation regularization. 3
1.3. Functionals on sets. 4
1.4. The family $\mathbf{n}_1^{loc}(M_{r,s})$ and its relationship to $\mathbf{m}_\epsilon^{loc}(F)$. 4
1.5. Acknowledgments. 5
2. Some useful definitions and notations. 6


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2.1. Quadratics. 7
3. The family $\Gamma_{r,s}(T)$, $(r,s) \in \mathbb{P}$. 7
4. A result on $\mathbf{n}_1^{loc}(M_{r,s})$, $(r,s) \in \mathbb{P}$. 8
5. The function $\Psi$ and the sets $Q$ and $N$. 9
6. Some results on $\Gamma_{r,s}(T)$, $(r,s) \in \mathbb{P}$. 12
6.1. A basic theorem. 13
6.2. An invariance theorem. 15
7. Two squares. 16
7.1. The relevant arc geometry. 16
7.2. The main theorem. 19
7.2.1. Some lemmas. 20
7.2.2. More lemmas. 21
7.3. Calculations. 22
7.3.1. Analysis of $\Phi_F$. 23
7.3.2. Analysis of $\Phi_G$. 24
7.4. Analysis of $\Phi_H$. 25
7.4.1. Determination of $\Psi$, $Q$, $q$ and $N$. 26
7.5. An analysis of three cases for $\gamma(y) = y^2/$, $y \in \mathbb{R}$. 28
8. Two circles. 28
8.1. Preliminaries. 29
8.2. The main theorem. 29
8.2.1. 30
8.2.2. The main lemma. 31
8.2.3. Completion of the proof of the main theorem. 32
8.3. Calculations. 32
8.3.1. Analysis of $\Phi_F$. 32
8.3.2. Analysis of $\Phi_S$. 33
8.3.3. Relationship between $\Phi_F$ and $\Phi_S$. 33
8.4. Determination of $\Psi$, $Q$, $q$ and $N$. 34
8.4.1. Case One. $l \geq \pi/2$. 34
8.4.2. Case Two. $1 < l < \pi/2$. 34
8.4.3. An analysis of three cases for $\gamma(y) = y^2/$, $y \in \mathbb{R}$. 35
8.4.4. Case Three. $l = 1$. 35
8.4.5. An analysis of three cases for $\gamma(y) = y^2/$, $y \in \mathbb{R}$. 36
References 36

## 1. Introduction.

1.1. **Total variation.** This work is based on the notion of the total variation of a locally summable function on $\mathbb{R}^2$, which we now define.

**Definition 1.1.** Suppose $f \in \mathbf{L}_1^{loc}(\mathbb{R}^2)$. We let

$$\mathbf{TV}(f) = \sup \left\{ \int f \,\mathbf{div}\, X \, d\mathcal{L}^2 : X \in \mathcal{X}(\mathbb{R}^2) \text{ and } |X| \leq 1 \right\}$$

and call this nonnegative extended real number the **total variation of** $f$; here $\mathcal{X}(\mathbb{R}^2)$ is the vector space of smooth compactly supported vector fields on $\mathbb{R}^2$ and and $\mathcal{L}^2$ is Lebesgue measure on $\mathbb{R}^2$.

In particular, if $f$ is continuously differentiable on $\mathbb{R}^2$ then

$$\mathbf{TV}(f) = \int |\nabla f| \, d\mathcal{L}^2. \tag{3}$$

Moreover, if $E$ a Lebesgue measurable subset of $\mathbb{R}^2$ with Lipschitz boundary then $\mathbf{TV}(E)$ equals the length of the boundary; *here and in what follows we frequently identify a subset $E$ of $\mathbb{R}^2$ with its indicator function $1_E$.*

1.2. **Total variation regularization.** We let

$$\mathcal{F}(\mathbb{R}^2) = \{f \in \mathbf{L}_1(\mathbb{R}^2) \cap \mathbf{L}_\infty(\mathbb{R}^2) : f \geq 0\}.$$

**Definition 1.2.** Suppose $F : \mathcal{F}(\mathbb{R}^2) \to \mathbb{R}$ and $0 < \epsilon < \infty$. We let

$$F_\epsilon(g) = \epsilon \mathbf{TV}(g) + F(g) \quad \text{for } g \in \mathcal{F}(\mathbb{R}^2)$$

and we let

$$\mathbf{m}_\epsilon^{loc}(F) = \{f \in \mathcal{F}(\mathbb{R}^2) : \mathbf{TV}(f) < \infty \text{ and } F_\epsilon(f) \leq F_\epsilon(g) \text{ for } g \in \mathbf{k}(f)\};$$

here $\mathbf{k}(f)$ is the set of $g \in \mathcal{F}(\mathbb{R}^2)$ such that, for some compact subset $K$ of $\mathbb{R}^2$, $g(x) = f(x)$ for $\mathcal{L}^2$ almost all $x \in \mathbb{R}^2 \sim K$.

*For the remainder of this paper, $S$ will be a compact subset of $\mathbb{R}^2$ with nonempty interior,*

$$0 < \epsilon < \infty$$

and

$$F(f) = \int_{\mathbb{R}^2} \gamma\left(f(x) - 1_S(x)\right) \, d\mathcal{L}^2 x \quad \text{for } f \in \mathcal{F}(\mathbb{R}^2)$$

*where*

$$\gamma : \mathbb{R} \to [0, \infty), \quad \gamma \text{ is convex}, \quad \gamma(0) = 0 \quad \text{and} \quad \gamma(y) > 0 \text{ if } y \in \mathbb{R} \sim \{0\}.$$

Of particular interest in the literature are when

$$\gamma(y) = \frac{y^2}{2} \quad \text{and when} \quad \gamma(y) = |y|;$$

the corresponding functionals $F_\epsilon$ were introduced in [ROF] and [CE], repsectively.

For example, $S$ could be a degraded binary image which we which we wish to denoise. In the context of denoising $F$ would be called a **fidelity** in that it is a measure of how much $f$ differs from $1_S$. The members of $\mathbf{m}_\epsilon(F)$ could be called **total variation regularizations of $S$ (with respect to the fidelity $F$ and smoothing parameter $\epsilon$).**

In the literature one oftens sets $\lambda = 1/\epsilon$ and studies

$$\lambda F_\epsilon(f) = \mathbf{TV}(f) + \lambda F(f), \quad f \in \mathcal{F}(\mathbb{R}^2).$$

For a very informative discussion of the use of total variation regularizations in the field of image processing see the Introduction of [CE]. We will not discuss image processing any further except to note that the notion of total variation regularization in image processing is useful for other purposes besides denoising.

In the paper [AW2] the family $\mathbf{m}_\epsilon^{loc}(F)$ was described in detail when $S$ is convex.

In this paper we will state and prove a number of theorems about $\mathbf{m}_\epsilon^{loc}(F)$. We will describe this family in detail when

$$S = ([0, 1] \times [0, -1]) \cup ([-1, 0] \times [0, 1])$$

and when

$$S = \{x \in \mathbb{R}^2 : |x - \mathbf{c}_+| \le 1\} \cup \{x \in \mathbb{R}^2 : |x - \mathbf{c}_-| \le 1\}$$

where, for some $l \in [1, \infty)$, $\mathbf{c}_\pm = (\pm l, 0)$. We believe these examples reveal a great deal about the nature of total variation regularizations. In addition, one can test computational schemes for total variation regularization against these examples.

1.3. **Functionals on sets.** It will be useful to extend the foregoing notions to functionals defined on sets, as follows.

We let

$$\mathcal{M}(\mathbb{R}^2) = \left\{D : D \subset \mathbb{R}^2 \text{ and } 1_D \in \mathcal{F}(\mathbb{R}^2)\right\};$$

thus a subset $D \in \mathcal{M}(\mathbb{R}^2)$ if and only $1_D \in \mathcal{F}(\mathbb{R}^2)$.

Suppose $M : \mathcal{M}(\mathbb{R}^2) \to \mathbb{R}$ and $0 < \zeta < \infty$. We let

$$M_\zeta(E) = \zeta \mathbf{TV}(E) + M(E) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2)$$

and we let

$$\mathbf{n}_\zeta^{loc}(M)$$

be the family of those $D \in \mathcal{M}(\mathbb{R}^2)$ such that $\mathbf{TV}(D) < \infty$ and $M_\zeta(D) \le M_\zeta(E)$ whenever $E \in \mathcal{M}(\Omega)$ and $1_E \in \mathbf{k}(1_D)$.

1.4. **The family $\mathbf{n}_1^{loc}(M_{r,s})$ and its relationship to $\mathbf{m}_\epsilon^{loc}(F)$.**

**Definition 1.3.** Let

$$\mathbb{P} = (0, \infty) \times (0, \infty).$$

**Definition 1.4.** For each $(r, s) \in \mathbb{P}$ we let

$$M_{r,s}(E) = -\frac{1}{r}\mathcal{L}^2(E \cap S) + \frac{1}{s}\mathcal{L}^2(E \sim S) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).$$

Let

$$\beta(y) = \limsup_{z \downarrow y} \frac{\gamma(z) - \gamma(y)}{z - y} \quad \text{for } y \in \mathbb{R}.$$

*Note that $\beta$ is nonincreasing and negative on $(-\infty, 0)$ and nondecreasing and positive on $(0, \infty)$.* Let

$$\mathbf{r}(y) = -\frac{\epsilon}{\beta(y-1)} \quad \text{and let} \quad \mathbf{s}(y) = \frac{\epsilon}{\beta(y)} \quad \text{for } 0 < y < 1;$$

note that $\mathbf{r}$ is nondecreasing and $\mathbf{s}$ is nonincreasing.

**Example 1.1.** If $\gamma(y) = |y|$ for $y \in \mathbb{R}$ then

$$\beta(y) = \begin{cases} -1 & \text{if } -\infty < y < 0, \\ 1 & \text{if } 0 \le y < \infty \end{cases}$$

so

$$\mathbf{range}\,(\mathbf{r}, \mathbf{s}) = \{(\epsilon, \epsilon)\}.$$

**Example 1.2.** If $\gamma(y) = y^2/2$ for $y \in \mathbb{R}$ then

$$\beta(y) = y \quad \text{for } y \in \mathbb{R}$$

so

$$\mathbf{range}\,(\mathbf{r}, \mathbf{s}) = \left\{(r, s) \in (\epsilon, \infty) \times (\epsilon, \infty) : \frac{1}{r} + \frac{1}{s} = \frac{1}{\epsilon}\right\}$$

which is a connected component of a hyperbola.

For each $y \in (0, \infty)$ we let

$$U_y(E) = \beta(y-1)\mathcal{L}^2(E \cap S) + \beta(y)\mathcal{L}^2(E \sim S) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2)$$

and note that

$$\frac{1}{\epsilon} U_y = M_{\mathbf{r}(y),\mathbf{s}(y)};$$

it follows that

$$\mathbf{n}_\epsilon^{loc}(U_y) = \mathbf{n}_1^{loc}(M_{r,s}). \tag{4}$$

**Theorem 1.1.** $\mathbf{n}_1^{loc}(M_{r,s}) \neq \emptyset$ for each $(r,s) \in \mathbb{P}$.

*Proof.* This follows from well known compactness theorems (see, for example,[AW1, Theorem 2.2]) and the lowersemicontinuity of $\mathbf{TV}(\cdot)$. □

**Definition 1.5.** Let

$$\Upsilon_\epsilon(F)$$

be the family of those $\upsilon$ such that

(i) $\upsilon$ is a function with domain $(0,1)$;
(ii) $\upsilon(y) \in \mathbf{n}_\epsilon^{loc}(M_{\mathbf{r}(y),\mathbf{s}(y)})$ for each $y \in (0,1)$;
(iii) $\upsilon(z) \subset \upsilon(y)$ whenever $0 < y < z < 1$.

For each $\upsilon \in \Upsilon_\epsilon(F)$ we let

$$f_\upsilon(x) = \mathcal{L}^1(\{y \in (0,1) : x \in \upsilon(y)\}).$$

**Remark 1.1.** Suppose $\upsilon \in \Upsilon_\epsilon(F)$. It follows from Tonelli's Theorem that

$$\mathcal{L}^2((\{f_\upsilon > y\} \sim \upsilon(y) \cup (\upsilon(y) \sim \{f_\upsilon > y\})) = 0 \quad \text{for } \mathcal{L}^1 \text{ almost all } y > 0$$

and it is obvious that

$$0 \leq f_\upsilon \leq 1.$$

In view of (4), the following result, which is the starting point for the results in this paper, follows directly from Theorems 1.6.1 and 1.6.2 of [AW1].

**Theorem 1.2.** We have $f \in \mathbf{m}_\epsilon^{loc}(F)$ if and only if $||f - f_\upsilon||_{\mathbf{L}_1(\mathbb{R}^2)} = 0$ for some $\upsilon \in \Upsilon_\epsilon(F)$.

*Thus if one can determine* $\mathbf{n}_1^{loc}(M_{r,s})$ *for* $(r,s) \in \mathbb{P}$ *one has determined* $\mathbf{m}_\epsilon^{loc}(F)$ *whenever* $S, F, \epsilon, \gamma$ *are as in Section 1.2.* In Sections 4 and 5 we will prove a number of interesting properties of $\mathbf{n}_1^{loc}(M_{r,s})$, $(r,s) \in \mathbb{P}$. In Section 7 we will determine $\mathbf{n}_1^{loc}(M_{r,s})$, $(r,s) \in \mathbb{P}$, when $S$ is as in (1) and in Section 8 we will determine $\mathbf{n}_1^{loc}(M_{r,s})$, $(r,s) \in \mathbb{P}$, when $S$ is as in (2).

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## 2. Some useful definitions and notations.

Whenever $a \in \mathbb{R}^2$ and $0 < r < \infty$ we let
$$\mathbf{U}(a,r) = \{x \in \mathbb{R}^2 : |x-a| < r\}, \quad \mathbf{B}(a,r) = \{x \in \mathbb{R}^2 : |x-a| \le r\}$$
and we let
$$\mathbf{C}(a,r) = \{x \in \mathbb{R}^2 : |x-a| = r\}.$$
We let
$$\mathbf{int}, \quad \mathbf{cl}, \quad \text{and} \quad \mathbf{bdry}$$
stand for "interior", "closure" and "boundary", respectively.

We let
$$\mathbf{spt}$$
stand for "support".

We let
$$\mathbf{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$$
and we let
$$\mathbf{e}_1 = (1,0) \in \mathbf{S}^1, \quad \mathbf{e}_2 = (0,1) \in \mathbf{S}^1.$$
We let
$$\mathcal{H}^1$$
be one dimensional Hausdorff measure on $\mathbb{R}^2$.

Suppose $A \subset \mathbb{R}^2$ and $a \in \mathbb{R}^2$. We let
$$\mathbf{Tan}(A,a) = \bigcap_{0<r<\infty} \mathbf{cl}\,\{t(x-a) : 0 < t < \infty \text{ and } x \in A \cap (\mathbf{B}(a,r) \sim \{a\})\}$$
if $a$ is an accumulation point of $A$ and we let $\mathbf{Tan}(A,a) = \{0\}$ if $a$ is an isolated point of $A$; and we let
$$\mathbf{Nor}(A,a) = \bigcap_{w \in \mathbf{Tan}(A,a)} \{v \in \mathbb{R}^2 : v \bullet w \le 0\}.$$
For each $\theta \in \mathbb{R}$ we let
$$\mathbf{u}(\theta) = (\cos\theta, \sin\theta).$$
Whenever $a, b \in \mathbb{R}^2$ we let
$$(a,b) = \{(1-t)a + tb : 0 < t < 1\} \quad \text{and we let} \quad [a,b] = \{(1-t)a + tb : 0 \le t \le 1\}.$$
Whenever $c \in \mathbb{R}^2$, $0 < r < \infty$, $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$ we let
$$\mathbf{a}(c,r,\alpha,\beta) = \{c + r\mathbf{u}(\theta) : \alpha < \theta < \beta\} \subset \mathbf{C}(a,r).$$
Whenever $0 < r < \infty$ we let
$$\mathbf{A}(r) = \{\mathbf{a}(c,r,\alpha,\beta) : c \in \mathbb{R}^2,\ 0 < r < \infty,\ \alpha < \beta,\ \alpha,\beta \in \mathbb{R} \text{ and } \beta - \alpha \le \pi\};$$
thus $\mathbf{A}(r)$ is the family of proper open arcs $A$ of circles of radius $r$ such that the length of $A$ does not exceed $\pi r$. Whenever $A \in \mathbf{A}(r)$ we let
$$\mathbf{ends}(A) = (\mathbf{cl}\,A) \sim A$$
and note that $\mathbf{ends}(A)$ contains exactly two points.

We let
$$L_1 = \{(t,0) : t \in \mathbb{R}\}; \quad L_2 = \{(0,t) : t \in \mathbb{R}\}$$
and we let
$$L_- = \{(t,t) : t \in \mathbb{R}\}; \quad L_- = \{(t,-t) : t \in \mathbb{R}\}.$$

We let $\iota$ be the identity map of $\mathbb{R}^2$, we let $\alpha(x) = -x$ for $x \in \mathbb{R}^2$, and we let $\rho_1, \rho_2, \rho_+, \rho_-$ be reflection across $L_1, L_2, L_+, L_-$, respectively.

**Definition 2.1.** For each $f \in \mathbf{L}_1^{loc}(\mathbb{R}^2)$ we let

$$[f]$$

be the generalized function corresponding to $f$. Let

$$\mathbf{L}(\mathbb{R}^2) = \{[f] : f \in \mathbf{L}_1^{loc}(\mathbb{R}^2)\}.$$

We partially order $\mathbf{L}(\mathbb{R}^2)$ by requiring that

$$[f] \preceq [g] \quad \Leftrightarrow \quad f(x) \leq g(x) \quad \text{for } \mathcal{L}^2 \text{ almost all } x.$$

We note that $\mathbf{L}(\mathbb{R}^2)$ is a complete lattice with respect to the partial order $\preceq$; in particular, we have

$$[f] \wedge [g] = [f \wedge g] \quad \text{and} \quad [f] \vee [g] = [f \vee g] \quad \text{for } f, g \in \mathbf{L}_1^{loc}(\mathbb{R}^2).$$

2.1. **Quadratics.** Suppose $I$ is an open interval,

$$a, b, c : I \to \mathbb{R}, \quad a \text{ never vanishes} \quad \text{and} \quad \Delta = b^2 - 4ac > 0.$$

Let

$$x_\pm = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{so that} \quad a x_\pm^2 + b x_\pm + c = 0 \quad \text{and} \quad x_- < x_+.$$

As the reader may easily verify, if $a, b, c$ are differentiable then so are $x_\pm$ and

$$\frac{\sqrt{\Delta}}{a} (x_\pm)' = \mp \left( \left( \frac{b}{a} \right)' x_\pm + \left( \frac{c}{a} \right)' \right).$$

## 3. The family $\Gamma_{r,s}(T)$, $(r, s) \in \mathbb{P}$.

**Definition 3.1.** Suppose $E \subset \mathbb{R}^2$ and $b \in \mathbf{bdry}\, E$. We say $b$ **is regular** if there are open intervals $I$ and $J$ containing 0; a continuously differentiable function $f : I \to J$; and an isometry $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(b) = 0$, $f(0) = 0$, $f'(0) = 0$, and

$$\Phi[E] \cap (I \times J) = \{(u, v) \in I \times J : v \leq f(u)\}.$$

**Definition 3.2.** Suppose $E$ and $T$ are compact subsets of $\mathbb{R}^2$. We let

$$\mathbf{cmp}(E, T)$$

be the family of connected components of $E \sim (\mathbf{bdry}\, T)$. We let

$$\mathbf{int}(E, T) = \{A \in \mathbf{cmp}(E, T) : A \subset \mathbf{int}\, T\},$$
$$\mathbf{ext}(E, T) = \{A \in \mathbf{cmp}(E, T) : A \subset \mathbb{R}^2 \sim T\}.$$

**Definition 3.3.** For each $(r, s) \in \mathbb{P}$ we let

$$\Gamma_{r,s}(T)$$

be the family of compact subsets $E$ of $\mathbb{R}^2$ such that the following conditions hold:

($\Gamma$0) each boundary point of $E$ is regular;
($\Gamma$1) $E$ is a subset of the convex hull of $T$;
($\Gamma$2) if $A \in \mathbf{int}(E, T)$ then $A \in \mathbf{A}(r)$ and the length of $A$ does not exceed $\pi r$;
($\Gamma$3) if $A \in \mathbf{int}(E, T)$ and $c$ is the center of the circle $C$ that contains $A$ there is an open subset $U$ of $\mathbf{int}\, T$ such that $A \subset U$ and $U \cap E = U \cap \mathbf{B}(c, r)$.
($\Gamma$4) if $A \in \mathbf{ext}(E, T)$ then $A \in \mathbf{A}(s)$ and the length of $A$ does not exceed $\pi s$;

(Γ5) if $A \in \mathbf{ext}(E,T)$ and $c$ is the center of the circle $C$ containing $A$ there is an open subset $U$ of $\mathbb{R}^2 \sim T$ such that $A \subset U$ and $U \cap E = U \sim \mathbf{U}(c,s)$.

Note that $\emptyset \in \Gamma_{r,s}(T)$ and that if $E \in \Gamma(r,s)$ then $\mathbf{spt}\,[E] = E$ by virtue of (Γ0).

The following Theorem provides the motivation for introducing $\Gamma_{r,s}(T)$. As will become evident later, it is false that $\Gamma_{r,s}(S) \subset \mathbf{n}_1^{loc}(M_{r,s})$.

**Theorem 3.1.** Suppose $(r,s) \in \mathbb{P}$ and $E \in \mathbf{n}_1^{loc}(M_{r,s})$. Then

$$[E] = [\mathbf{spt}\,[E]] \quad \text{and} \quad \mathbf{spt}\,[E] \in \Gamma_{r,s}(S).$$

*Proof.* This follows from [AW1, Section 8]. □

## 4. A result on $\mathbf{n}_1^{loc}(M_{r,s})$, $(r,s) \in \mathbb{P}$.

**Theorem 4.1.** Suppose $(r,s) \in \mathbb{P}$ and $\mathcal{A}$ is a nonempty subset of $\{[D] : D \in \mathbf{n}_1^{loc}(M_{r,s})\}$. Then there are $D, E \in \mathbf{n}_1^{loc}(M_{r,s})$ such that

$$[D] = \sup \mathcal{A} \quad \text{and} \quad [E] = \inf \mathcal{A};$$

here inf and sup are with respect to the lattice $\mathbf{L}(\mathbb{R}^2)$.

**Remark 4.1.** In other words, $\{[D] : D \in \mathbf{n}_1^{loc}(M_{r,s})\}$ is a complete sublattice of $\mathbf{L}(\mathbb{R}^2)$.

**Remark 4.2.** We prove this by a natural modification of the proof of [AW1, Theorem 1.7] which deals with the case $r = 1 = s$.

Theorem 1.7 of [AW1] is incorrect as stated because it does not take into account of the obvious fact that if, in the notation used there, $D \in \mathbf{n}_\epsilon^{loc}(N_S)$, $E \in \mathcal{M}(\Omega)$ and $\mathcal{L}^2((D \sim E) \cup (E \sim D)) = 0$ then $E \in \mathbf{n}_\epsilon^{loc}(N_S)$; a correct version is the above statement with $r = 1 = s$.

*Proof.* Let

$$\zeta(y) = \begin{cases} \frac{y}{s} & \text{if } 0 \le y < \infty, \\ -\frac{y}{r} & \text{if } -\infty < y < 0 \end{cases}$$

and let

$$G(g) = \int \zeta(g(x) - 1_S(x))\, d\mathcal{L}^2 x \quad \text{for } g \in \mathcal{F}(\mathbb{R}^2).$$

For each $y \in (0,\infty)$ let $U_y$ be as in [AW1, Theorem 1.5] and note that

$$U_y = \begin{cases} 0 & \text{if } 1 \le y < \infty, \\ M_{r,s} & \text{if } 0 < y < 1. \end{cases}$$

To prove the Theorem proceed as in in the proof of [AW1, Theorem 1.7], keeping in mind that $\zeta$ is convex. □

The following Theorem follows directly follows from the previous Theorem and Theorem 1.2.

**Theorem 4.2.** $\{[f] : f \in \mathbf{m}_\epsilon^{loc}(F)\}$ is a complete sublattice of $\mathbf{L}(\mathbb{R}^2)$.

## 5. The function $\Psi$ and the sets $Q$ and $N$.

**Definition 5.1.** Keeping in mind Theorem 1.1 and the fact that $(M_{r,s})_1(\emptyset) = 0$ we define

$$\Psi : \mathbb{P} \to (-\infty, 0]$$

by letting

$$\Psi(r,s) = (M_{r,s})_1(D) \quad \text{whenever } (r,s) \in \mathbb{P} \text{ and } D \in \mathbf{n}_1^{loc}(M_{r,s}).$$

We let

$$Q = \{(r,s) \in \mathbb{P} : \text{for some } D,\ D \in \mathbf{n}_1^{loc}(M_{r,s}) \text{ and } \mathcal{L}^2(D) > 0\}.$$

$$N = \{(r,s) \in \mathbb{P}; \text{there are } D_1, D_2 \in \mathbf{n}_1^{loc}(M_{r,s}) \text{ such that } [D_1] \neq [D_2]\}$$

Thus for $(r,s) \in \mathbb{P}$ we have $(r,s) \in Q$ if and only if $(M_{r,s})_1$ has a nontrivial minimizer and $(r,s) \in N$ if and only if $(M_{r,s})_1$ has at least two essentially distinct minimizers.

*Throughout this Section, let $T$ is the convex hull of $S$.*

Evidently,

(5)
$$(r_i,s_i) \in \mathbb{P},\ i = 1,2,\ r_1 \le r_2 \text{ and } s_2 \le s_1 \ \Rightarrow\ M_{r_1,s_1} \le M_{r_2,s_2} \text{ and } (M_{r_1,s_1})_1 \le (M_{r_2,s_2})_1 .$$

Using $S$ and $\emptyset$ as comparisons we find that

$$-\frac{1}{r}\mathcal{L}^2(S) \le \Psi(r,s) \le 0.$$

**Proposition 5.1.** Suppose $(r,s) \in \mathbb{P}$ and $E \in \mathcal{M}(\mathbb{R}^2)$. Then

$$(M_{r,s})_1(E \cap T) \le (M_{r,s})_1(E).$$

*Proof.* See [AW1, Proposition 10.2]. □

**Corollary 5.1.** Suppose $(r,s) \in \mathbb{P}$ and $D \in \mathbf{n}_1^{loc}(M_{r,s})$. Then $\mathbf{spt}\,[D] \subset T$.

**Proposition 5.2.** $\Psi$ is locally Lipschitzian; in fact,

$$|\Psi(r_1,s_1) - \Psi(r_2,s_2)| \le \left( \left| \frac{1}{r_1} - \frac{1}{r_2} \right| + \left| \frac{1}{s_1} - \frac{1}{s_2} \right| \right) \mathcal{L}^2(T) \tag{6}$$

whenever $(r_i,s_i) \in \mathbb{P}$, $i = 1,2$.

Moreover, if $(r_i,s_i) \in \mathbb{P}$, $i = 1,2$, we have

$$r_1 \le r_2 \text{ and } s_2 \le s_1 \ \Rightarrow\ \Psi(r_1,s_1) \le \Psi(r_2,s_2) \tag{7}$$

as well as

$$(r_2,s_2) \in Q,\ r_1 < r_2 \text{ and } s_2 < s_1 \ \Rightarrow\ \Psi(r_1,s_1) < \Psi(r_2,s_2). \tag{8}$$

*Proof.* Suppose $(r_i,s_i) \in \mathbb{P}$, $i = 1,2$. Let $\delta > 0$. Keeping in mind the preceding Proposition, for each $i = 1,2$ we choose $E_i \in \mathcal{M}(\mathbb{R}^2)$ such that

$$E_i \subset T \quad \text{and} \quad (M_{r_i,s_i})_1(E_i) \le \Psi(r_i,s_i) + \delta.$$

Suppose $\{i,j\}=\{1,2\}$. Then

$$\begin{aligned}\Psi(r_i,s_i) &\le \left(M_{r_i,s_i}\right)_1(E_j)\\ &= \left(M_{r_j,s_j}\right)_1(E_j)+\left(\frac{1}{r_j}-\frac{1}{r_i}\right)\mathcal{L}^2(E_j\cap S)+\left(\frac{1}{s_i}-\frac{1}{s_j}\right)\mathcal{L}^2(E_j\sim S)\\ &\le \Psi(r_j,s_j)+\delta+\left(\frac{1}{r_j}-\frac{1}{r_i}\right)\mathcal{L}^2(E_j\cap S)+\left(\frac{1}{s_i}-\frac{1}{s_j}\right)\mathcal{L}^2(E_j\sim S)\end{aligned}$$

which, owing to the arbitrariness of $\delta$, implies

$$\Psi(r_i,s_i)\le\Psi(r_j,s_j)+\left(\frac{1}{r_j}-\frac{1}{r_i}\right)\mathcal{L}^2(E_j\cap S)+\left(\frac{1}{s_i}-\frac{1}{s_j}\right)\mathcal{L}^2(E_j\sim S). \tag{9}$$

(9) immediately implies (6) as well as (9) by taking $i=1$ and $j=2$.

Suppose $(r_2,s_2)\in Q$, $r_1<r_2$ and $s_2<s_1$. Since $(r_2,s_2)\in Q$ we can require $E_2\in\mathbf{n}_1^{loc}(M_{r_2,s_2,S})$ and $\mathcal{L}^2(E_2)>0$. Let $i=1$ and $j=2$ in (9) we find that (8) holds since at least one of $\mathcal{L}^2(E_2\cap S)$ and $\mathcal{L}^2(E_2\sim S)$ is positive. □

**Theorem 5.1.** Suppose $(r,s)\in\mathbb{P}$, $D\in\mathbf{n}_1^{loc}(M_{r,s})$ and $\mathcal{L}^2(D)>0$. Then

$$\mathbf{TV}(D)\ge\frac{2}{e}\min\{r,s\}\quad\text{and}\quad\mathcal{L}^2(D)\ge\frac{2}{e^2}\min\{r,s\}.$$

*Proof.* It is immediate that $D\in\mathcal{C}_\lambda(\mathbb{R}^2)$ where $\lambda=\max\{1/r,1/s\}$ and where $\mathcal{C}_\lambda(\mathbb{R}^2)$ is as in [AW1, 1.5]. The asserted inequalities now follow from [AW1, Theorem 5.4]. □

**Proposition 5.3.** Suppose $(r_i,s_i)\in\mathbb{P}$, $i=1,2$, $r_1\le r_2$ and $s_2\le s_1$. Then

$$(r_2,s_2)\in Q\ \Rightarrow (r_1,s_1)\in Q.$$

Moreover, if $D_i\in\mathbf{n}_1^{loc}(M_{r_i,s_i})$, $i=1,2$, then

$$r_1<r_2\ \ \Rightarrow\ \ [(D_2\cap S]\preceq[D_1\cap S]\quad\text{and}\quad s_2<s_1\ \ \Rightarrow\ \ [D_2\sim S]\preceq[D_1\sim S].$$

*Proof.* From (5)

(10)

$$D_2\in\mathbf{n}_1^{loc}(M_{r_2,s_2})\ \Rightarrow\ \left(M_{r_1,s_1}\right)_1(D_2)\le\left(M_{r_2,s_2}\right)_1(D_2)=\Psi(r_2,s_2)\le\left(M_{r_2,s_2}\right)_1(\emptyset)=0.$$

If $(r_2,s_2)\in Q$ there is $D_2\in\mathbf{n}_1^{loc}(M_{r_2,s_2})$ such that $\mathcal{L}^2(D_2)>0$ so implies $(r_1,s_1)\in Q$.

Suppose $D_i\in\mathbf{n}_1^{loc}(M_{r_i,s_i})$, $i=1,2$. By [AW1, Proposition 9.2] we infer that

$$M_{r_2,s_2}(D_2\sim D_1)\le M_{r_1,s_1}(D_2\sim D_1)$$

which amounts to

$$\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\mathcal{L}^2((D_2\sim D_1)\cap S)+\left(\frac{1}{s_2}-\frac{1}{s_1}\right)\mathcal{L}^2((D_2\sim D_1)\sim S)\le 0;$$

the final assertion of the Proposition directly follows. □

**Theorem 5.2.** There is a nondecreasing function

$$q:(0,\infty)\to(0,\infty)$$

such that

$$Q=\{(r,s)\in\mathbb{P}:r\le q(s)\}.$$

Moreover, $Q$ is closed relative to $\mathbb{P}$ and

$$\{(r,s) \in \mathbb{P} : \Psi(r,s) < 0\} = \{(r,s) \in \mathbb{P} : r < \sup\{q(\sigma) : 0 < \sigma < s\}\}.$$

*Proof.* Suppose $(r,s)$ is a sequence in $Q$ with limit $(r_\infty, s_\infty) \in \mathbb{P}$. Let

$$\lambda = \sup\left\{\max\left\{\frac{1}{r_\nu}, \frac{1}{s_\nu}\right\} : \nu = 1,2,3,\ldots\right\} < \infty.$$

For each $\nu = 1,2,3,\ldots$ choose $E_\nu \in \mathbf{n}_1^{loc}(M_{r_\nu,s_\nu})$ such that $\mathcal{L}^2(E_\nu) > 0$ and note that $E_\nu \in \mathcal{C}_\lambda(\mathbb{R}^2)$ where $\mathcal{C}_\lambda(\mathbb{R}^2)$ is as in [AW1, 1.5]. Passing to a subsequence if necessary we use [AW1, Theorem 2.2, Theorem 5.2] to obtain $D \in \mathcal{M}(\mathbb{R}^2)$ such that $\mathcal{L}^2((E_\nu \sim D) \cup (D \sim E_\nu)) \to 0$ and $(M_{r_\nu,s_\nu})_1(E_\nu) \to (M_{r_\infty,s_\infty})_1(D)$ as $\nu \to \infty$. Since $\Psi(r_\nu, s_\nu) = (M_{r_\nu,s_\nu})_1(E_\nu)$, $\nu = 1,2,3,\ldots$, and since $\Psi$ is continuous we find that $\Psi(r_\infty, s_\infty) = (M_{r_\nu,s_\nu})_1(D)$. From Theorem 5.1 we infer that $\mathcal{L}^2(D) > 0$. Thus $(r_\infty, s_\infty) \in Q$ so $Q$ is closed relative to $\mathbb{P}$.

For each $s \in (0,\infty)$ let $I(s) = \{r : (r,s) \in Q\}$ and let $q(s) = \sup I(s) \in \{-\infty\} \cup [0,\infty]$.

Suppose $s \in (0,\infty)$. If $0 < r < s\mathcal{L}^2(S)/(s\mathbf{TV}(T) + \mathcal{L}^2(T))$ then

$$(M_{r,s})_1(T) = \mathbf{TV}(T) - \frac{1}{r}\mathcal{L}^2(S) + \frac{1}{s}\mathcal{L}^2(T \sim S) < 0$$

so $r \in I(s)$ and, therefore, $I(s) \neq \emptyset$. Suppose $0 < r_1 < r_2 \in I(s)$. For any $s_1 \in (s,\infty)$ we infer from (8) that $\Psi(s_1, r_1) < \Psi(r_2, s) \leq 0$ so $(r_1, s) \in Q$. Since $Q$ is closed we conclude that $r_1 \in I(s)$. Thus $I(s)$ is an interval. Let

$$J = \{r \in (0,\infty) : (2/e)\min\{r,s\} - \mathcal{L}^2(S)/r > 0\},$$

note that $J$ is nonempty and choose $r \in J$. Were it the case that $(r,s) \in Q$ there would be $D \in \mathbf{n}_1^{loc}(M_{r,s})$ such that $\mathcal{L}^2(D) > 0$. But by Theorem 5.1 we would have

$$0 = (M_{r,s})_1(\emptyset) \geq (M_{r,s})_1(D) \geq \mathbf{TV}(D) - \frac{1}{r}\mathcal{L}^2(D \cap S) \geq \frac{2}{e}\min\{r,s\} - \frac{1}{r}\mathcal{L}^2(S) > 0.$$

It follows that $I(s)$ is bounded. Let $q(s) = \max I(s)$.

Suppose $0 < s_1 < s_2 < \infty$. If $0 < r_1 < r_2 = q(s_2)$ then $\Psi(r_1, s_1) < \Psi(r_2, s_2) \leq 0$ by (8) so $(r_1, s_1) \in Q$ so $r_1 \in I(s_1)$; that is, $I(s_2) \subset I(s_1)$ so $q$ is nondecreasing.

Finally, if $0 < s < \infty$ and $0 < r < \sup\{q(\sigma) : 0 < \sigma < s\}$ there is $(r_2, s_2) \in Q$ such that $r < r_2$ and $s_2 < s$ so that, by 8, $\Psi(r,s) < \Psi(r_2, s_2) \leq 0$. □

**Remark 5.1.** I do not know if the boundary of $Q$ can contain horizontal segments.

**Definition 5.2.** Suppose $(r,s) \in \mathbb{P}$. Keeping in mind Theorem 4.1

$$\underline{D}_{r,s} \in \mathbf{n}_1^{loc}(M_{r,s}) \quad \text{and} \quad \overline{D}_{r,s} \in \mathbf{n}_1^{loc}(M_{r,s})$$

be requiring that $\underline{D}_{r,s} = \mathbf{spt}\,[\underline{D}_{r,s}]$, $\overline{D}_{r,s} = \mathbf{spt}\,[\overline{D}_{r,s}]$,

$$[\underline{D}_{r,s}] = \inf\{[E] : E \in \mathbf{n}_1^{loc}(M_{r,s})\}, \quad \text{and} \quad [\overline{D}_{r,s}] = \sup\{[E] : E \in \mathbf{n}_1^{loc}(M_{r,s})\};$$

here inf and sup are with respect to the lattice $\mathbf{L}(\mathbb{R}^2)$.

Whenever $0 < \eta < \infty$ we let

$$N_\eta = \{(r,s) \in N : \mathcal{L}^2\left(\overline{D}_{r,s} \sim \underline{D}_{r,s}\right) \geq \eta\}.$$

**Proposition 5.4.** Suppose $0 < \eta < \infty$. Then $N_\eta$ is closed.

*Proof.* Argue as we did in the beginning of the proof of Theorem 5.2. □

**Proposition 5.5.** Suppose $0 < \eta < \infty$,

$$X = \{x \in \mathbb{R}^2 : x_1 \le 0 \text{ and } x_2 \le 0\} \cup \{x \in \mathbb{R}^2 : x_1 \ge 0 \text{ and } x_2 \ge 0\}$$

and $(r, s)$ is an accumulation point of $N_\eta$.

Then there is $\delta > 0$ such that

$$N_\eta \cap \mathbf{U}((r, s), \delta) \subset (r, s) + X;$$

in particular,

$$\mathbf{Tan}(N_\eta, (r, s)) \subset X.$$

*Proof.* If the Proposition were false we could choose an increasing sequence $\rho$ in $(0, r)$ and a decreasing sequence $\sigma$ in $(s, \infty)$ such that $(\rho_\nu, \sigma_\nu) \in N_\eta$ and $(\rho_\nu, \sigma_\nu) \to (r, s)$ as $\nu \to \infty$.

It follows from Proposition 5.3 that the sets

$$\overline{D}_{\rho_\nu,\sigma_\nu} \sim \underline{D}_{\rho_\nu,\sigma_\nu}, \quad \nu = 1, 2, 3, \dots$$

are disjointed. But each is a subset of $T$ and has area at least $\eta$; this is impossible. □

**Corollary 5.2.** $N$ is a Borel set which is countably $(\mathcal{H}^1, 1)$-rectifiable in the sense of [FE, 3.2.14].

## 6. Some results on $\Gamma_{r,s}(T)$, $(r, s) \in \mathbb{P}$.

*For the remainder of this section we fix $(r, s) \in \mathbb{P}$ and a compact subset $T$ of $\mathbb{R}^2$.*

**Lemma 6.1.** Suppose $E \in \Gamma(r, s)$, $A \in \mathbf{cmp}(E, T)$, $C$ is the circle containing $A$, $Y$ is a line or a circle and $\mathbf{ends}(A) \subset Y$. Then $C$ meets $Y$ transversely.

*Proof.* Since the length of $C$ exceeds the length of $A$, $C$ must meet both connected components of $\mathbb{R}^2 \sim Y$. □

**Proposition 6.1.** Suppose $E \in \Gamma_{r,s}(T)$; $A \in \mathbf{cmp}(E, T)$; $b \in \mathbf{ends}(A)$; $u$ and $v$ are such that

$$\mathbf{S}^1 \cap \mathbf{Nor}(E, b) = \{u\}; \quad \mathbf{S}^1 \cap \mathbf{Tan}(A, b) = \{v\}; \quad H = -\mathbf{Nor}(A, b).$$

If $A \in \mathbf{int}(E, T)$ then

$$A \subset \mathbf{C}(b - ru, r)$$

and there is an open subset $U$ of $\mathbb{R}^2$ such that $b \in U$ and

$$E \cap H \cap U = \mathbf{B}(b - ru, r) \cap H \cap U.$$

If $A \in \mathbf{ext}(E, T)$ then

$$A \subset \mathbf{C}(b + su, s)$$

and there is an open subset $U$ of $\mathbb{R}^2$ such that $b \in U$ and

$$E \cap H \cap U = (H \sim \mathbf{U}(b + su, s)) \cap U.$$

*Proof.* Since $b$ is a regular boundary point of $E$ and $A \subset \mathbf{bdry}\, E$ we find that $\mathbf{Tan}(A, b) \subset \mathbf{Tan}(\mathbf{bdry}\, E, b)$ and $u \bullet v = 0$. Thus there are $\eta, J, f, g, \Phi$ such that $0 < \eta < \infty$; $J$ is an open interval in $\mathbb{R}$ containing 0; $f : (-\eta, \eta) \to J$; $g : (0, \eta) \to J$; $f$ and $g$ are continuously differentiable; $f(0) = 0 = \lim_{t \downarrow 0} g(t)$; $f'(0) = 0 = \lim_{t \downarrow 0} g'(t)$; $\Phi$ is a an isometry of $\mathbb{R}^2$;

$$\Phi[E] \cap ((-\eta, \eta) \times J) = \{(t, u) \in (-\eta, \eta) \times J : u \le f(t)\}$$

and

$$\Phi[A] \cap ((0,\eta) \times J) = g.$$

Since $A \subset \mathbf{bdry}\, E$ we find that $g \subset f$. We leave it to the reader to use (Γ3) and (Γ5) to supply the remaining details of the proof. □

**Proposition 6.2.** Suppose $b \in \mathbf{bdry}\, T$ and

$$\mathcal{A} = \{A \in \mathbf{cmp}(E,T) : b \in \mathbf{ends}(A)\}.$$

Then

(i) $\mathcal{A}$ has at most two members;

(ii) if $b \in \mathbf{bdry}\, E$ and *either* $b$ is an isolated point of $\mathbf{bdry}\, T$ *or* $b$ is an accumulation point of $\mathbf{bdry}\, T$ and

$$\mathbf{Tan}(\mathbf{bdry}\, E, b) \cap \mathbf{Tan}(\mathbf{bdry}\, T, b) = \{0\}$$

then $\mathcal{A}$ has exactly two members;

(iii) if $b \in \mathbf{bdry}\, E$, $b$ is a regular boundary point of $T$ and

$$\mathbf{Tan}(\mathbf{bdry}\, E, b) \cap \mathbf{Tan}(\mathbf{bdry}\, T, b) = \{0\}$$

then there are $A \in \mathbf{int}(E,T)$ and $B \in \mathbf{ext}(E,T)$ such that $\mathcal{A} = \{A, B\}$; moreover, if $c, d$ are the centers of the circles containing $A, B$, respectively, then $c \neq d$ and the points $b, c, d$ are collinear.

*Proof.* The previous Proposition directly implies (i).

Suppose the hypotheses of (ii) hold. Let $I, J, f, \Phi$ be as in (Γ0). Shrinking $I$ and $J$ if necessary we may assume that $\Phi[\mathbf{bdry}\, T] \cap f = \{(0,0)\}$. Let $I_+ = \{t \in I : t > 0\}$ and let $I_- = \{t \in I : t < 0\}$. Then $\Phi^{-1}[I_+]$ and $\Phi^{-1}[I_-]$ are connected subsets of $\mathbb{R}^2 \sim \mathbf{bdry}\, T$ so there are $A_+$ and $A_-$ in $\mathbf{cmp}(E,T)$ such that $\Phi^{-1}[I_+] \subset A_+$ and $\Phi^{-1}[I_-] \subset A_-$. Given the length restriction on each of $A_+$ and $A_-$ we find that $A_+ \neq A_-$. Thus (ii) holds.

We leave it as a simple exercise for the reader to prove (iii) making use of (Γ3) and (Γ5). □

6.1. **A basic theorem.** The proof of the following Theorem is an elementary though tedious exercise in plane geometry.

**Theorem 6.1.** Suppose

(i) $E \in \Gamma_{r,s}(T)$ and $Y$ is a line or a circle;

(ii) $G$ is the set of $a \in \mathbb{R}^2$ such that *either* $a \notin \mathbf{bdry}\, T$ *or* $a \in \mathbf{bdry}\, T$ and there is an open subset $W$ of $\mathbb{R}^2$ such that $a \in W$ and $W \cap \mathbf{bdry}\, T = W \cap Y$;

(iii) $X$ is a connected component of $G \cap \mathbf{bdry}\, E$ and $X \cap \mathbf{bdry}\, T \neq \emptyset$;

(iv)

$$P = \{a \in X \cap Y : \mathbf{Tan}(X,a) = \mathbf{Tan}(Y,a)\};$$

$$Q = \{a \in X \cap Y : \mathbf{Tan}(X,a) \cap \mathbf{Tan}(Y,a) = \{0\}\};$$

$$\mathcal{R}_1 = \{A \in \mathbf{cmp}(E,T) : \mathbf{card}\,(P \cap \mathbf{ends}(A)) = 1 \text{ and } \mathbf{card}\,(\mathbf{ends}(A) \sim X) = 1\};$$

$$\mathcal{R}_2 = \{A \in \mathbf{cmp}(E,T) : \mathbf{card}\,(Q \cap \mathbf{ends}(A)) = 1 \text{ and } \mathbf{card}\,(\mathbf{ends}(A) \sim X) = 1\};$$

$$\mathcal{R}_3 = \{A \in \mathbf{cmp}(E,T) : \mathbf{card}\,(Q \cap \mathbf{ends}(A)) = 2\};$$

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3.$$

Then

(v) $G$ is open, $G \cap \mathbf{bdry}\, T = G \cap Y$ and $X \cap \mathbf{bdry}\, T = X \cap Y$;

(vi) $X$ is a relatively open subset of $\mathbf{bdry}\,E$; $X \cap \mathbf{bdry}\,T = P \cup Q$; $X \sim \mathbf{bdry}\,T = \cup\mathcal{R}$;
(vii) $P$ is connected, $Q$ is finite, $\mathbf{card}\,\mathcal{R}_1 + \mathbf{card}\,\mathcal{R}_2 \leq 2$ and $\mathcal{R}_3$ is finite.
(viii) if $P \neq \emptyset$ then
  (a) $Q = \emptyset$, $\mathcal{R}_2 \cup \mathcal{R}_3 = \emptyset$;
  (b) $X$ is homeomorphic to a line;
  (c) $1 \leq \mathbf{card}\,\mathcal{R}_1 \leq 2$ and $\{P, \cup\mathcal{R}_1\}$ is a partition of $X$;
(ix) if $Q \neq \emptyset$ then
  (a) $P = \emptyset$ and $R_1 = \emptyset$;
  (b) *either* $1 \leq \mathbf{card}\,\mathcal{R}_2 \leq 2$, $X$ is homeomorphic to a line and $\{Q, \cup\mathcal{R}_2, \cup\mathcal{R}_3\}$ is a partition of $X$ *or* $\mathbf{card}\,\mathcal{R}_2 = 0$, $X$ is homeomorphic to a circle, $\mathbf{card}\,Q \geq 5$ and $\{Q, \cup\mathcal{R}_3\}$ is a partition of $X$;
  (c) if $A, B \in \mathcal{R}_2 \cup \mathcal{R}_3$, *either* $A, B \in \mathbf{int}(E, T)$ *or* $A, B \in \mathbf{ext}(E, T)$ and $C, D$ are the circles containing $A, B$, respectively, and $\mathbf{M}$ is the set of rigid motions of $\mathbb{R}^2$ which carry $Y$ into $Y$ and which carry the center of $C$ to the center of $D$ then $\mathbf{M}$ has exactly two members;
  (d) if $A, B, C, D, \mathbf{M}$ are as in (c) and $\mathbf{ends}(A) \cup \mathbf{ends}(B) \subset Q$ then $\sigma[A] = B$ for $\sigma \in \mathbf{M}$;
  (e) if $A, B, C, D, \mathbf{M}$ are as in (c) and $Y$ is a line then one member of $\mathbf{M}$ is the translation which carries the center of $C$ to the center of $D$ and the other is reflection about the perpendicular bisector of the segment joining the centers of $C$ and $D$;
  (f) if $A, B, C, D, \mathbf{M}$ are as in (c) and $Y$ is a circle then one member of $\mathbf{M}$ is the rotation about the center of $Y$ that carries the center of $C$ to the center of $D$ and the other is reflection about the perpendicular bisector of the segment joining the centers of $C$ and $D$.

*Proof.* (v) is obvious. Since $G$ is open we infer that $G \cap \mathbf{bdry}\,E$ is open relative to $\mathbf{bdry}\,E$. Since $\mathbf{bdry}\,E$ is locally connected we infer that $X$ is a relatively open subset $\mathbf{bdry}\,E$. In particular, $X$ is homeomorphic to a line or a circle. If $a \in X \cap Y$ then both $\mathbf{Tan}(X, a)$ and $\mathbf{Tan}(Y, a)$ are lines; this implies that $X \cap Y = P \cup Q$. If $x \in X \sim Y$ then there is $A \in \mathbf{ext}(E, T)$ such that $x \in A$. Since $X$ is connected and $X \cap \mathbf{bdry}\,T \neq \emptyset$ there is $a \in Y \cap \mathbf{ends}(A)$. Let $C$ be the circle containing $A$. If $C$ meets $Y$ tangentially then $A \in \mathcal{R}_1$ since the length of $A$ is less than the length of $C$. If $C$ meets $Y$ transversely then $A \in \mathcal{R}_2 \cup \mathcal{R}_3$. Thus (vi) holds and we have

$$A \in \mathcal{R} \;\Rightarrow\; Y \cap \mathbf{ends}(A) \neq \emptyset. \tag{11}$$

**Definition 6.1.** Suppose $\mathcal{C}$ is a subfamily of $\mathcal{R}$. We let

$$\mathbf{spt}\,\mathcal{C} = X \cap (\cup\{\mathbf{cl}\,A : A \in \mathcal{C}\}).$$

We say $\mathcal{C}$ is a **chain** if
(a) $\mathbf{spt}\,\mathcal{C}$ is connected.
(b) $\mathcal{C}$ is finite;
(c) if $A \in \mathcal{C}$ and $C$ is the circle containing $A$ then $C$ meets $Y$ transversely.

**Lemma 6.2.** Suppose $\mathcal{C}$ is a chain. Then there exist a positive integer $N$ and functions

$$A : \{1, \ldots, N\} \to \mathcal{R} \quad \text{and} \quad a : \{0, 1, \ldots, N\} \to X$$

such that $A$ is univalent with range $\mathcal{C}$ and such that

$$\mathbf{ends}(A_i) = \{a_{i-1}, a_i\} \quad \text{whenever i=1,\ldots,N.}$$

Moreover, $a_i \in Q$ if $0 < i < N$.

*Proof.* This is a straightforward consequence of ($\Gamma$0), Proposition 6.1 and the connectedness of $\mathbf{spt}\,\mathcal{C}$. $\square$

**Lemma 6.3.** Suppose

(i) $\mathcal{C}$ is a chain;
(ii) $N, A, a$ are as in Lemma 6.2 and $N = 3$;
(iii) for each $i = 1, 2, 3$, $C_i$ is the circle containing $A_i$ and $B_i$ is the connected component of $C_i \sim Y$ such that $A_i \subset B_i$;
(iv) $\rho_2$ is reflection across the perpendicular bisector of $[a_1, a_2]$.

Then
$$\rho_2[B_1] = B_3 \quad \text{and} \quad \rho_2[C_1] = C_3.$$

*Proof.* Since $X$ is homeomorphic to a line or a circle we find that $\{a_1, a_2\} \subset X$. We infer from Proposition 6.1 that $C_2$ meets $Y$ transversely. One now invokes Proposition 6.2. $\square$

**Lemma 6.4.** Any chain is a subfamily of a unique maximal chain.

*Proof.* Suppose $\mathcal{C}$ is a chain, $N, A, a$ are as in Lemma 6.2 and $N \geq 3$. It follows from the preceding Lemma $A_i$ is congruent to $A_j$ whenever $i, j \in \{2, N-1\}$ and $i - j$ is even. Since the length of $\mathbf{bdry}\,E$ and therefore $X$ is finite we infer that $\{\mathbf{card}\,\mathcal{D} : \mathcal{D} \text{ is a chain}\}$ is bounded.

Suppose $\mathcal{C}$ is a chain and $N, A, a$ are as in Lemma 6.2. Suppose $\{i, j\} = \{0, N\}$, $a_i \in X$ and $C$ is the circle containing $A_i$. Since $C$ meets $Y$ transversely there is $B \in \mathbf{ext}(E, T)$ such that $B \neq A_i$. In case $B \in \mathcal{C}$ then $B = A_j$ and $\mathcal{C}$ is maximal; otherwise $\mathcal{C} \cup \{B\}$ is a chain with one more member that $\mathcal{C}$. In case $\{a_0, a_N\} \cap X = \emptyset$ we find that $\mathcal{C}$ is maximal. $\square$

Suppose $Q \neq \emptyset$. Then there is a nontrivial maximal chain $\mathcal{C}$. Since $X$ is connected we have $X = \mathbf{spt}\,\mathcal{C}$. In particular, $Q$ is finite. We leave it to the reader to use the foregoing Lemmas to prove that (a)-(f) of (ix) hold.

Suppose $Q = \emptyset$. Suppose $A \in \mathbf{ext}(E, T)$ and $C$ is the circle containing $A$. Since $X \cap \mathbf{bdry}\,T \neq \emptyset$ and $X$ is connected there is $a \in Y \cap \mathbf{ends}(A)$. Since $Q = \emptyset$ we find that $C$ meets $Y$ tangentially. Since the length of $A$ is less than the length of $C$ we have $\{a\} = Y \cap \mathbf{ends}(A)$ so $A \in \mathcal{R}_3$. Since, in this case, $X$ is homeomorphic to a line we find that $P = X \sim \cup\mathcal{R}_3$ is connected and that (a)-(c) of (viii) hold. $\square$

### 6.2. **An invariance theorem.**

**Theorem 6.2.** Suppose $E \in \mathbf{n}_1^{loc}(M_{r,s})$, $b \in \mathbf{bdry}\,E \sim \mathbf{bdry}\,S$, $H = \mathbf{Tan}(E, b)$, $\sigma$ is a rigid motion of $\mathbb{R}^2$, $\sigma[S] = S$ and $\sigma(b) = b$.

If $b \in \mathbf{int}\,S$ then $\sigma[b + H] = b + H$.

If $b \notin S$ then *either* $\sigma[b + H] = b + H$ *or* $\sigma[b + H] = -(b + H)$.

*Proof.* Translating by $-b$ if necessary, we may assume without loss of generality that $b = 0$ and $\sigma$ is linear. We have
$$[1_{E \cup \sigma[E]}] = [1_E \vee 1_{\sigma[E]}] = [1_E] \vee [1_{\sigma[E]}]$$

so Theorem 4.1 implies $E \cup \sigma[E] \in \mathbf{n}_1^{loc}(M_{r,s})$. Keeping in mind Theorem 3.1 we find that

$$E \cup \sigma[E] = \mathbf{spt}\,[E] \cup \mathbf{spt}\,[\sigma[E]] = \mathbf{spt}\,[E \cup \sigma[E]] \in \Gamma_{r,s}(S).$$

This implies that *either* (i) $0 \in \mathbf{int}\, E \cup \sigma[E]$ and $\mathbf{Tan}(E \cup \sigma[E], 0) = \mathbb{R}^2$ *or* (ii) $0 \in \mathbf{bdry}\,(E \cup \sigma[E])$ and $\mathbf{Tan}(E \cup \sigma[E], 0)$ is a closed halfspace containing 0.

We also have $\mathbf{Tan}(E \cup \sigma[E], 0) = H \cup \sigma[H]$. It follows that $\sigma[H] = -H$ if (i) holds and that $\sigma[H] = H$ if (ii) holds.

In case $0 \in \mathbf{int}\, S$, (i) cannot hold in view of ($\Gamma$3) and ($\Gamma$0). □

## 7. Two squares.

Let

$$S = ([0,1] \times [0,-1]) \cup ([0,-1] \times [0,1]).$$

We shall determine $\mathbf{n}_1^{loc}(M_{r,s})$, $(r,s) \in \mathbb{P}$.

Let

$$\Sigma = \{\iota, \mu, \rho_+, \rho_-\}$$

and note that $\Sigma$ is the group of rigig motions of $\mathbb{R}^2$ which carry $S$ into itself.

Let

$$T = [0,1] \times [0,-1].$$

Let

$$V = \{(0,0), (1,0), (1,-1), (0,-1), (0,1), (-1,1), (-1,0)\};$$

thus $V$ is the set of the seven vertices of the polygon $S$. Let

$$I = \{(x_1, 0) : 0 < x_1 < 1\} \quad \text{and let} \quad J = \{(1, x_2) : -1 < x_2 < 0\}.$$

Let

$$\mathcal{E} = \{\sigma[I] : \sigma \in \Sigma\} \cup \{\sigma[J] : \sigma \in \Sigma\};$$

thus $\mathcal{E}$ is the set of the eight open edges of the polygon $\mathbf{bdry}\, S$.

### 7.1. The relevant arc geometry.

**Definition 7.1.** For $0 < s < \infty$ and $\pi/4 < \theta \le \pi/2$ we let

$$\begin{aligned}
\mathbf{c}_1(s,\theta) &= s \sin\theta (1,1);\\
\mathbf{a}_1(s,\theta) &= \mathbf{a}(\mathbf{c}_1(s,\theta), s, 5\pi/4 - (\theta - \pi/4), 5\pi/4 + (\theta - \pi/4));\\
\mathbf{p}_1(s,\theta) &= (s(\sin\theta - \cos\theta, 0);\\
\mathbf{q}_1(s,\theta) &= (0, s(\sin\theta - \cos\theta))
\end{aligned}$$

and note that $\mathbf{a}_1(s,\theta) \in \mathbf{A}(s)$ and $\mathbf{ends}(\mathbf{a}_1(s,\theta)) = \{\mathbf{p}_1(s,\theta), \mathbf{q}_1(s,\theta)\}$.

For $0 < r < \infty$ and $\pi/4 < \theta \le \pi/2$ we let

$$\begin{aligned}
\mathbf{c}_2(r,\theta) &= (1 - r, -r\sin\theta);\\
\mathbf{a}_2(r,\theta) &= \mathbf{a}(\mathbf{c}_2(r,\theta), r, 0, \theta);\\
\mathbf{p}_2(r,\theta) &= (1, -r\sin\theta);\\
\mathbf{q}_2(r,\theta) &= (1 - r(1 - \cos\theta), 0)
\end{aligned}$$

and note that $\mathbf{a}_2(r,\theta) \in \mathbf{A}(r)$ and $\mathbf{ends}(\mathbf{a}_2(r,\theta)) = \{\mathbf{p}_2(r,\theta), \mathbf{q}_2(r,\theta)\}$.

For $0 < r < \infty$ we let

$$\begin{aligned}\mathbf{c}_3(r) &= (1-r, -1+r);\\ \mathbf{a}_3(r) &= \mathbf{a}(\mathbf{c}_3(r), r, -\pi/2, 0);\\ \mathbf{p}_3(r) &= (1-r, -1);\\ \mathbf{q}_3(r) &= (1, -1+r)\end{aligned}$$

and note that $\mathbf{a}_3(r) \in \mathbf{A}(r)$ and $\mathbf{ends}(\mathbf{a}_3(r)) = \{\mathbf{p}_3(r), \mathbf{q}_3(r)\}$.

For $0 < r < \infty$ we let

$$\begin{aligned}\mathbf{c}_4(r) &= (r, -r);\\ \mathbf{a}_4(r) &= \mathbf{a}(\mathbf{c}_4(r), r, \pi/2, \pi);\\ \mathbf{p}_4(r) &= (0, -r);\\ \mathbf{q}_4(r) &= (r, 0)\end{aligned}$$

and note that $\mathbf{a}_4(r) \in \mathbf{A}(r)$ and $\mathbf{ends}(\mathbf{a}_4(r)) = \{\mathbf{p}_4(r), \mathbf{q}_4(r)\}$.

**Definition 7.2.** We let

$$w(r,s,\theta) = r(1-\cos\theta) + s(\sin\theta - \cos\theta) \quad \text{for } ((r,s),\theta) \in \mathbb{P} \times \mathbb{R}$$

and we let

$$\begin{aligned}V'_F &= \{((r,s),\theta) \in \mathbb{P} \times (\pi/4, \pi/2] : w(r,s,\theta) = 1\}\\ &= \{((r,s),\theta) \in \mathbb{P} \times (\pi/4, \pi/2] : \mathbf{p}_1(s,\theta) = \mathbf{q}_2(r,\theta)\},\\ W'_F &= \{(r,s) \in \mathbb{P} : (1-\sqrt{2}/2)r < 1 \le r+s\},\\ X'_F &= \{(r,\theta) : \pi/4 < \theta \le \pi/2 \text{ and } 0 < r < 1/(1-\sqrt{2}/2)\}.\end{aligned}$$

**Proposition 7.1.** There is one and only one function

$$\Theta : W'_F(\pi/4, \pi/2]$$

such that

$$((r,s),\theta) \in V'_F \;\Leftrightarrow\; (r,s) \in W'_F \text{ and } \theta = \Theta(r,s).$$

Moreover,

$$(r,s_i) \in W'_F,\ i = 1,2, \text{ and } s_1 < s_2 \;\Rightarrow\; \Theta(r,s_1) > \Theta(r,s_2).$$

*Proof.* Suppose $(r,s) \in W'_F$. Then

$$\frac{\partial}{\partial\theta} w(r,s,\theta) = r\sin\theta + s(\cos\theta + \sin\theta) > 0 \quad \text{for } \theta \in (\pi/4, \pi/2]$$

as well as

$$w(r,s,\pi/4) = r(1-\sqrt{2}/2) < 1 \quad \text{and} \quad w(r,s,\pi/2) = r+s \ge 1.$$

The final assertion follows by differentiating $w(r,s,\Theta(r,s))$ with respect $s$ for $(r,s)$ interior to $W'_F$. □

**Definition 7.3.** We let

$$\zeta(r,\theta) = \frac{1-r(1-\cos\theta)}{\sin\theta - \cos\theta} \quad \text{and} \quad Z(r,\theta) = (r, \zeta(r,\theta)) \quad \text{for } (r,\theta) \in X'_F.$$

**Corollary 7.1.** $Z$ and

$$W'_F \ni (r,s) \mapsto (r, \Theta(r,s))$$

are inverse to one another. Moreover, for any $r \in (0, 1/(1-\sqrt{2}/2))$,

$$(\pi/4, \pi/2] \ni \theta \mapsto \zeta(r,\theta) \quad \text{is decreasing.}$$

**Definition 7.4.** Let

$$\rho(\theta) = \frac{1}{1+\sin\theta} \quad \text{for } \pi/4 < \theta \le \pi/2$$

and let

$$\begin{aligned} W_F &= \{(r,s) \in W'_F : r \le 1/(1+\sin\Theta(r,s))\}, \\ W_G &= \{(r,s) \in \mathbb{P} : r+s \le 1 \text{ and } r \le 1/2\}, \\ W_H &= \{(r,s) \in \mathbb{P} : r \le 1/2\}. \end{aligned}$$

If we set

$$X_F = \{(r,\theta) \in X_F : r \le \rho(\theta)\}$$

we find that $X_F \subset X'_F$ and

$$W_F = \{Z(r,\theta) : (r,\theta) \in X_F\} \subset W'_F.$$

Let

$$b = \{(\rho(\theta), Z(\rho(\theta), \theta)) : \pi/4 < \theta \le \pi/2\}.$$

With the help of Maple we find that $b$ is an increasing function carrying $(1/2, \infty)$ onto $[1/2, 1/(1+\sqrt{2}/2))$. It is then easy to see that the boundary of $W_F$ is

$$\{(0,s) : 1 < s < \infty\} \cup \{(r, 1-r) : 0 < r < 1/2\} \cup \{(s, b(s) : 1/2 < s < \infty\}.$$

A plot of the boundary of $W_F$ appears in Figure ??.

**Remark 7.1.** Thus

$$W_F \cap W_G = \emptyset, \quad W_G \subset W_H, \quad W_H \cap \{(r,s) \in \mathbb{P} : r+s \ge 1\} \subset W_F.$$

For each $(r,s) \in W_F$ we let

$$F_{r,s}$$

be the compact subset of $\mathbb{R}^2$ whose boundary is the union of the the sets

$$\sigma[\mathbf{a}_1(s, \Theta(r,s))],\ \sigma \in \{\iota, \alpha\};$$
$$\sigma[\mathbf{a}_2(r,s,\Theta(r,s)) \cup [\mathbf{p}_2(s,\Theta(r,s)), \mathbf{q}_3(r)]],\ \sigma \in \Sigma;$$
$$\sigma[\mathbf{a}_3(r)],\ \sigma \in \{\iota, \alpha\}.$$

For each $(r,s) \in W_G$ we let

$$G_{r,s}$$

be the compact subset of $\mathbb{R}^2$ whose boundary equals the union of the sets

$$\sigma[\mathbf{a}_1(s, \pi/2)], \quad \sigma \in \{\iota, \alpha\};$$
$$\sigma[\mathbf{p}_1(s,\pi/2), \mathbf{q}_2(r,\pi/2)] \cup \mathbf{a}_2(r,\pi/2) \cup [\mathbf{p}_2(r,\pi/2), \mathbf{q}_3(r)]], \quad \sigma \in \Sigma,$$
$$\sigma[\mathbf{a}_3(r)], \quad \sigma \in \{\iota, \alpha\}.$$

For each $r \in W_H$ we let

$$I_r$$

be the compact subset of $\mathbb{R}^2$ whose boundary equals the convex hull of the union of the arcs

$$\mathbf{a}_3(r), \quad \sigma[\mathbf{a}_2(r,\pi/2)], \quad \sigma \in \{\iota, \rho_-\}, \quad \mathbf{a}_4(r).$$

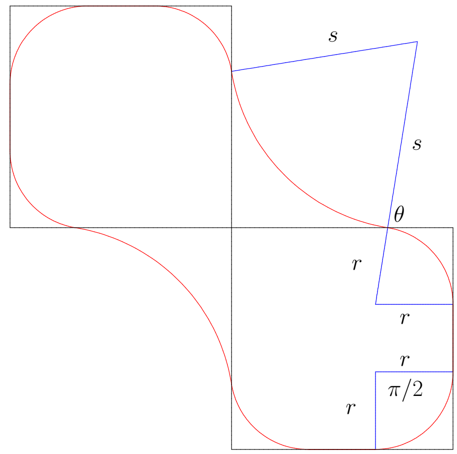


FIGURE 1. $F_{r,s}$ with $r = .35$, $s = .85$ and $\theta = \Theta(r, s)$.

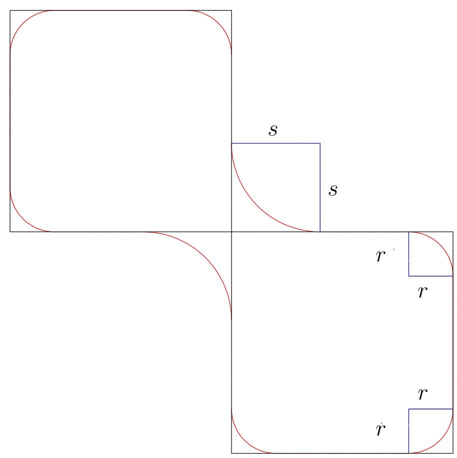


FIGURE 2. $G_{r,s}$ with $r = .2$ and $s = .4$.

For each $r \in W_H$ we let

$$H_r = I_r \cup \alpha[I_r].$$

7.2. **The main theorem.**

**Theorem 7.1.** Suppose $(r, s) \in \mathbb{P}$ and $E \in \mathbf{n}_1^{loc}(M_{r,s})$. Then *either*

$$(r, s) \in W_F \text{ and } [E] \in \{0, [F_{r,s}]\}$$

*or*

$$(r, s) \in W_G \text{ and } [E] \in \{0, [G_{r,s}]\}$$

*or*

$$(r, s) \in W_H \text{ and } [E] \in \{0, [H_r], [I_r], [\alpha[I_r]]\}.$$

**Remark 7.2.** More than one of these cases can occur.

The remainder of this section is devoted to the proof of this Theorem.

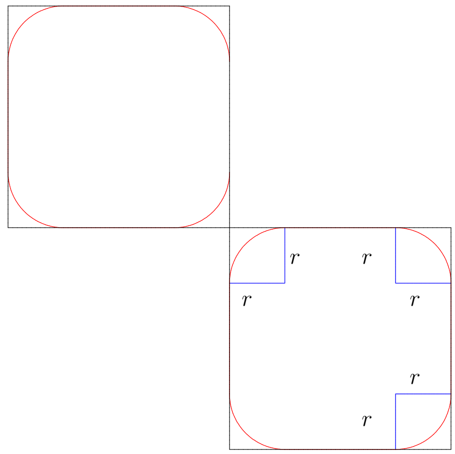


FIGURE 3. $H_{r,s}$ with $r = .2$.

7.2.1. *Some lemmas.* We suppose throughout this subsection that $(r, s) \in \mathbb{P}$ and $E \in \Gamma_{r,s}(S)$.

Let

$$M = \{x \in \mathbb{R}^2 : x \bullet \mathbf{e}_1 = 1\}.$$

**Lemma 7.1.** $(V \sim \{(0,0)\}) \cap \mathbf{bdry}\, E = \emptyset$.

*Proof.* This follows directly from $(\Gamma 0)$ and $(\Gamma 1)$. □

**Lemma 7.2.** Suppose $K \in \mathcal{E}$ and $A \in \mathbf{cmp}(E, S)$. Then

$$\mathbf{ends}(A) \not\subset \mathbf{cl}\, K.$$

*Proof.* Replacing $E$ by $\sigma[E]$ for some $\sigma \in \Sigma$ if necessary, we may assume that $K = I$ or $K = J$.

Suppose, contrary to the Lemma, $\mathbf{ends}(A) \subset \mathbf{cl}\, J$. By Proposition 6.1 the circle containing $A$ meets the line $M$ transversely. This is incompatible with $(\Gamma 0)$ and $(\Gamma 1)$.

Suppose, contrary to the Lemma, $\mathbf{ends}(A) \subset \mathbf{cl}\, I$. Since $(1, 0) \notin \mathbf{bdry}\, E$ and since $\mathbf{ends}(A)$ has two members there is $a \in I \cap \mathbf{ends}(A)$ From Proposition 6.1 we infer that that the circle containing $A$ meets $L_1$ transversely. Let $Y = L_1$, let $G = (\mathbb{R}^2 \sim \mathbf{bdry}\, S) \cup I$, let $X$ be the connected component of $a$ in $G \cap \mathbf{bdry}\, E$ and note that $G$ satisfies (ii) of Theorem 6.1. Let $P, Q$, etc., be as in Theorem 6.1. Since $a \in Q$ and since $Q$ is finite there is $q \in Q$ such that

$$q \bullet \mathbf{e}_1 \geq x \bullet \mathbf{e}_1 \quad \text{for } x \in Q. \tag{12}$$

By Theorem 6.1 there are $B', B'' \in \mathbf{cmp}(E,)$ such that $q \in \mathbf{ends}(B') \cap \mathbf{ends}(B'')$ and

$$x \bullet \mathbf{e}_1 < q \bullet \mathbf{e}_1 \quad \text{for } x \in B'.$$

Let $C', C''$, $c', c''$ be the circles containing $B', B''$ and their centers, respectively.

Suppose $B' \in \mathbf{int}(E, S)$. Then $c' \bullet \mathbf{e}_2 \geq 0$ which implies $c'' \bullet \mathbf{e}_1 \leq 0$. This implies that if $q''$ is such that $\mathbf{ends}(B'') = \{q, q''\}$ then $q'' \bullet \mathbf{e}_1 > q \bullet \mathbf{e}_1$. Since $q'' \notin \mathbf{bdry}\, E$ by Lemma 7.1 and since $C''$ meets $L_1$ transversely by Proposition 6.1 we conclude that $q'' \in Q$ which is incompatible with (12).

Suppose $B' \in \mathbf{ext}(E,S)$. Then $c' \bullet \mathbf{e}_2 \le 0$ which implies $c'' \bullet \mathbf{e}_1 \ge 0$. Since $C''$ cannot meet the line $M$ transversely by ($\Gamma$0) and ($\Gamma$1) we find that $(1,0) \notin C''$. But this is incompatible with Lemma 7.1 □

**Lemma 7.3.** Suppose $J \cap \mathbf{bdry}\, E \neq \emptyset$. There is $\theta \in (\pi/4, \pi/2]$ such that $r(1+\sin\theta) \le 1$ and

$$\mathbf{a}_2(r,\theta) \cup [\mathbf{p}_2(r,\theta), \mathbf{q}_3(r)] \cup \mathbf{a}_3(r) \subset \mathbf{bdry}\, E. \tag{13}$$

*Proof.* Let $Y = M$ , let $G = (\mathbb{R}^2 \sim \mathbf{bdry}\, S) \cup J$ and note that $G$ satisfies (ii) of Theorem 6.1. Let $X$ be the connected component of $G \cap \mathbf{bdry}\, E$ containing some point of $J \cap \mathbf{bdry}\, E$. Let $P, Q$, etc., be as in Theorem 6.1. Then, by Theorem 6.1, $P \neq \emptyset$, $P$ is connected and $Q = \emptyset$. Keeping in mind Lemma 7.1 we infer there exist $a, b \in J$ such that $a \bullet \mathbf{e}_2 > b \bullet \mathbf{e}_2$ and $P = [a,b]$. It follows that $\mathcal{R}_3 = \{A, B\}$ where $A, B \in \mathbf{int}(E,S)$, $a \in \mathbf{ends}(A)$ and $b \in \mathbf{ends}(B)$. We note that the lengths of $A$ and $B$ cannot exceed $\pi r$ and leave the remaining details of the proof to the reader. □

**Theorem 7.2.** Suppose $E \subset S$. Then $r \le 1/2$ and

$$E \in \{\emptyset, H_r, I_r, \alpha[I_r]\}.$$

*Proof.* Were it the case $0 \in \mathbf{bdry}\, E$ we would have $\mathbf{Tan}(E,0)) \subset \mathbf{Tan}(S,0)$. This is impossible since $\mathbf{Tan}(E,0)$ is a halfspace whose boundary contains 0. Thus $0 \in E$ which together with Lemma 7.1 implies $V \cap \mathbf{bdry}\, E = \emptyset$. We note that the lengths of any member of $\mathbf{cmp}(E,S)$ cannot exceed $\pi r$ and leave the remaining details of the proof to the reader. □

7.2.2. *More lemmas.* Now suppose $(r,s) \in \mathbb{P}$, $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\,[E]$. By Theorem 3.1 we have that $E \in \Gamma_{r,s}(S)$.

**Lemma 7.4.** Suppose $A \in \mathbf{int}(E,S)$ and $A \cap L_- \neq \emptyset$. Then $\{A, \alpha[A]\} \cap \{\mathbf{a}_3(r), \mathbf{a}_4(r)\} \neq \emptyset$.

*Proof.* We may suppose without loss of generality that $A \subset T$. Let $c$ be the center of the circle containing $A$. Then $c \in L_-$ by Theorem 6.2 and, since the length of $A$ cannot exceed $\pi r$, *either* (i) $\mathbf{cl}\, A$ meets both $J$ and $\rho_-[J]$ *or* (ii) $\mathbf{cl}\, A$ meets both $I$ and $\rho_-[I]$.

In case (i) holds then $C$ meets both $J$ and $\rho_-[J]$ tangentially since $E$ is a subset of the convex hull of $S$. Since the length of $A$ does not exceed $\pi r$ we find that $A = \mathbf{a}_3(r)$.

So suppose (ii) holds. Were it the case that $C$ met $I$ transversely there would by $B \in \mathbf{ext}(E,S)$ such that $\mathbf{ends}(A) \cap \mathbf{ends}(B) \neq \emptyset$. By Lemma 7.2 $B$ meets $L_+$. By Theorem 6.2 the center $c$ of the circle $D$ containing $B$ lies on $L_+$. Using ($\Gamma$3) and ($\Gamma$4) we infer that $\mathbf{a}_1(s,\theta)$ for some $\theta \in (\pi/4, \pi/2]$. But this forces the length of $A$ to equal $(2(\pi/2 - \theta) + \pi/2)r > \pi r$. Thus $C$ meets $I$ tangentially. Since the length of $A$ cannot exceed $\pi r$ we find that $A = \mathbf{a}_4(r)$. □

**Lemma 7.5.** Suppose $E \cap \mathbb{P} \neq \emptyset$. Then *either* $(r,s) \in W_F$ and $E = F_{r,s}$ *or* $(r,s) \in W_G$ and $E = G_{r,s}$.

*Proof.* Replacing $E$ by $\alpha[E]$ if necessary, we may choose $A \in \mathbf{ext}(E,S)$. By Lemma 7.2 there is $a \in I \cap \mathbf{ends}(A)$. Let $C$ be the circle containing $A$ and let $c$ be its center. From Theorem 6.2 we infer that $c \in L_+$.

**Case One.** $C$ **meets** $L_1$ **tangentially at** $a$**.** Since the length of $A$ does not exceed $\pi s$ and since $c \in L_+$ we find that $A = \mathbf{a}_1(s, \pi/2)$. Letting $Y = L_1$, let $G = (\mathbb{R}^2 \sim \mathbf{bdry}\, S) \cup I$, let $X$ be the connected component of $G \cap \mathbf{bdry}\, E$ which contains $a$ and let $P, Q, \mathcal{R}$, etc., be as in Theorem 6.1. It follows from Theorem 6.1 that $P \neq \emptyset$. Since $(1, 0) \notin \mathbf{bdry}\, E$ by Lemma 7.1 there is $B \neq A$ such that $\mathcal{R}_3 = \{A, B\}$. Were it the case that $B \in \mathbf{ext}(E, S)$ the length of $B$ would equal $3\pi 2/2$ which we have excluded; thus $B \in \mathbf{int}(E, S)$. Since $B \cap L_- = \emptyset$ by Lemma 7.4 we infer from Lemma 7.2 that $J \cap \mathbf{ends}(B) \neq \emptyset$. We now use Lemma 7.3 to see that ?? holds.

**Case Two.** $C$ **meets** $L_1$ **transversely at** $a$**.** In this case there is $B \in \mathbf{int}(E, S)$ such that $a \in \mathbf{ends}(B)$. Were it the case that $B$ met $L_-$ we would have $B = \mathbf{a}_4(r)$ by Lemma 7.4. But this would force the length of $A$ to equal $3\pi s/2$. Thus $J \cap \mathbf{ends}(B) \neq \emptyset$. By Lemma 7.3, there is $\theta \in (\pi/4, \pi/2]$ such that (13 holds. Since $\mathbf{c}_2(r, \theta), a, c$ are collinear we infer that $A = \mathbf{a}_1(s, \theta)$. □

7.3. **Calculations.** We set

$$\Phi_F(r,s) = rs\,(M_{r,s})_1\,(F_{r,s}) = rs(\mathbf{TV}(F_{r,s}) + M_{r,s}(F_{r,s})), \quad (r,s) \in W_F;$$
$$\Phi_G(r,s) = rs\,(M_{r,s})_1\,(G_{r,s}) rs(\mathbf{TV}(G_{r,s}) + M_{r,s}(G_{r,s})), \quad (r,s) \in W_G;$$
$$\Phi_H(r,s) = rs\,(M_{r,s})_1\,(H_r) rs(\mathbf{TV}(H_r) + M_{r,s}(H_r)), \quad (r,s) \in W_H$$

and proceed to calculate these functions.

We have

$$\mathcal{H}^1(\mathbf{a}_1(s,\theta)) = (2\theta - \pi/2)s \quad \text{whenever } 0 < s < \infty \text{ and } \pi/4 < \theta \le \pi/2;$$
$$\mathcal{H}^1(\mathbf{a}_2(r,\theta)) = r\sin\theta \quad \text{whenever } 0 < r < \infty \text{ and } \pi/4 < \theta \le \pi/2;$$
$$\mathcal{H}^1(\mathbf{a}_3(r,\theta)) = \frac{\pi}{4} r \quad \text{whenever } 0 < r < \infty.$$

For $(r,s) \in W_F$ we let $\theta = \Theta(r,s)$ and note that

$$|\mathbf{p}_2(r,\theta) - \mathbf{q}_3(r)| = 1 - r(1 + \sin\theta).$$

For $(r,s) \in W_G$ we have

$$|\mathbf{p}_1(s,\pi/2) - \mathbf{q}_2(r,\pi/2)| = 1 - (r+s).$$

For $0 < s < \infty$ and $\pi/4 < \theta \le \pi/2$ we let

$$R_1(s,\theta) = \mathbb{P} \cap \{(1-t)\mathbf{c}_1(s,\theta) + tx : 1 < t < \infty \text{ and } x \in \mathbf{a}_1(s,\theta)\}$$

and we calculate

$$\mathcal{L}^2(R_1(s,\theta)) = s^2 - (s\cos\theta)(s\sin\theta) - \frac{1}{2}(2\theta - \pi/2)s^2 = s^2\left(1 + \frac{\pi}{4} - \theta - \sin\theta\cos\theta\right).$$

For $0 < r < \infty$ and $\pi/4 < \theta \le \pi/2$ we let

$$R_2(r,\theta) = T \cap \{(1-t)\mathbf{c}_2(r,\theta) + tx : 0 < t < \infty \text{ and } x \in \mathbf{a}_2(r,\theta)\}$$

and we calculate

$$\mathcal{L}^2(R_2(r,\theta)) = r(r\sin\theta) - \frac{r^2}{2}\theta - \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{r^2}{2}(\sin\theta(2 - \cos\theta) - \theta).$$

For $0 < r < \infty$ we let

$$R_3(r) = T \cap \{(1-t)\mathbf{c}_3(r) + tx : 1 < t < \infty \text{ and } x \in \mathbf{a}_3(r)\}$$

and we calculate

$$\mathcal{L}^2(R_3(r)) = r^2 - \frac{\pi}{2}r^2 = \left(1 - \frac{\pi}{2}\right)r^2.$$

7.3.1. *Analysis of* $\Phi_F$. For $(r, s, \theta) \in \mathbb{R}^3$ we let

$$\begin{aligned}\phi_F(r,s,\theta) &= -4\,r^2\sin\theta\cos\theta - 4\,rs\sin\theta\cos\theta + 4\,rs\theta - rs\pi \\ &\qquad + 4\,\theta r^2 + 8\,r - 4\,r^2 + \pi\,r^2 - 4 - 4\,rs\sin^2\theta.\end{aligned} \tag{14}$$

Suppose $(r, s) \in W_F$ and $\theta = \Theta(r, s)$. Then

$$\begin{aligned}\mathbf{TV}(F_{r,s}) &= 2\mathcal{H}^1(\mathbf{a}_1(s,\theta)) + 4\mathcal{H}^1(\mathbf{a}_2(r,\theta)) + 4|\mathbf{p}_2(r,\theta) - \mathbf{q}_3(r)| + 2\mathcal{H}^1(\mathbf{a}_3(r));\\ \mathcal{L}^2(F_{r,s}\cap S) &= 2(1 - 2\mathcal{L}^2(R_2(r,\theta)) - \mathcal{L}^2(R_3(r));\\ \mathcal{L}^2(F_{r,s}\sim S) &= 2(1-\mathcal{L}^2(R_1(s,\theta)))\end{aligned}$$

so that

$$\Phi_F(r,s) = \frac{s}{2}\phi_F(r,s,\theta) \tag{15}$$

With the help of Maple we obtain

$$\phi_F(Z(r,\theta),\theta) = \frac{\zeta(r,\theta)}{2(\sin\theta-\cos\theta)}a_F(\theta)r^2 + b_F(\theta)r + c_F(\theta)$$

for $(r,\theta) \in \mathbb{R} \times (\pi/4, \pi/2]$ where for $\theta \in (\pi/4, \pi/2]$ we have set

$$\begin{aligned}a_F(\theta) &= 4\sin\theta\cos\theta - 2\pi\cos\theta + 4\cos\theta + \pi\sin\theta\\ &\qquad + 4\theta\sin\theta - 4\sin\theta - 4\theta + \pi - \sin^2\theta;\\ b_F(\theta) &= -8\cos\theta + 4 - 4\cos^2\theta + 8\sin\theta - 4\sin\theta\cos\theta - \pi + 4\theta;\\ c_F(\theta) &= -4\sin\theta + 4\cos\theta.\end{aligned}$$

With the help of Maple we find that

$$a_F < 0, \quad b_F > 0, \quad c_F < 0, \quad \Delta_F > b_F^2$$

where we have set

$$\Delta_F(\theta) = b_F(\theta)^2 - 4a_F(\theta)c_F(\theta) \quad \text{for } \theta \in (\pi/4, \pi/2].$$

Let

$$\rho_{F,\pm}(\theta) = \frac{-b_F(\theta) \pm \sqrt{\Delta_F(\theta)}}{2a_F(\theta)} \quad \text{for } \theta \in (\pi/4, \pi/2].$$

Since $a_F < 0$ we find that for any $r \in \mathbb{R}$ we have

$$\begin{aligned}\phi_F(Z(r,\theta)) < 0 &\Leftrightarrow r \in (-\infty, \rho_{F,-}(\theta)) \cup (\rho_{F,+}(\theta), \infty),\\ \phi_F(r,\zeta(r,\theta),\theta) = 0 &\Leftrightarrow r = \rho_{F,\pm}(\theta),\\ \phi_F(r,\zeta(r,\theta),\theta) > 0 &\Leftrightarrow r \in (\rho_{F,-}(\theta)), \rho_{F,+}(\theta)).\end{aligned}$$

For any $\theta \in (\pi/4, \pi/2]$ we have

$$\phi_F(0,\zeta(0,\theta),\theta) = -2\frac{1}{\sin\theta - \cos\theta} < 0.$$

as well as

$$\phi_F(\rho(\theta),\zeta(\rho(\theta),\theta),\theta) = \frac{\zeta(\rho(\theta),\theta))}{(\sin\theta-\cos\theta)\rho(\theta)^2}(4\theta\sin\theta - \pi\cos\theta) > 0$$

for $\theta \in (\pi/4, \pi/2]$. It follows that

$$0 < \rho_{F,-}(\theta) < \rho(\theta) < \rho_{F,+}(\theta) \quad \text{for } \pi/4 < \theta < \pi/2$$

so that for any $(r,\theta) \in X_F$ we have

$$\Phi_F(Z(r,\theta)) < 0 \;\Leftrightarrow\; r < \rho_{F,-}(\theta);$$
$$\Phi_F(Z(r,\theta)) = 0 \;\Leftrightarrow\; \rho_{F,-}(\theta) = r;$$
$$\Phi_F(Z(r,\theta)) > 0 \;\Leftrightarrow\; \rho_{F,-}(\theta) < r.$$

Plotting $\rho_{F,-}$ as well as its derivative using Maple we find that

(16) $$\rho_{F,-} \quad \text{is decreasing.}$$

Let

$$r_{F,-} = \rho_{F,-}(\pi/2), \quad r_{F,+} = \lim_{\theta\downarrow\pi/4} \rho_{F,-}(\theta).$$

We use Maple to obtain

$$r_{F,-} = \frac{\pi + 12 \pm \sqrt{\pi^2 + 56\,\pi + 16}}{16 - 4\pi}, \quad r_{F,+} = \frac{2(1+\sqrt{2}) - 2\sqrt{2+\pi}}{4 - \sqrt{2}(\pi - 2)}.$$

Let

$$\sigma_{F,-}(\theta) = \zeta(\rho_{F,-}(\theta), \theta) \quad \text{for } \theta \in (\pi/4, \pi/2].$$

Plotting $\sigma_{F,-}$ using Maple we find that

(17) $$\sigma_{F,-} \quad \text{is decreasing.}$$

Let

$$s_{F,-} = \sigma_{F,-}(\pi/2), \quad s_{F,+} = \lim_{\theta\downarrow\pi/4} \sigma_{F,-}(\theta).$$

We use Maple to obtain

$$s_{F,-} = \frac{5\pi - 4 + \sqrt{\pi^2 + 56\,\pi + 16}}{16 - 4\pi}, \quad s_{F,+} = \infty.$$

It follows that

$$q_F = \{(\sigma_{F,-}(\theta), \rho_{F,-}(\theta)) : \theta \in (\pi/4, \pi/2]\}$$

is an increasing function on $[s_{F,-}, \infty)$ with range equal $[r_{F,-}, r_{F,+})$ such that

(18)
$$\{(r,s) \in W_F : \Phi_F(r,s) < 0\} = \{(r,s) : s_{F,-} \le s < \infty,\ 0 < r < q_F(s) \text{ and } r+s \ge 1\}.$$

7.3.2. *Analysis of* $\Phi_G$. Suppose $(r,s) \in W_G$. Then

$$\begin{aligned}\mathbf{TV}(G_{r,s}) &= 2\mathcal{H}^1(\mathbf{a}_1(s,\pi/2)) + 4|\mathbf{p}_1(s,\pi/2) - \mathbf{q}_2(r,\pi/2)| \\ &\quad + 4\mathcal{H}^1(\mathbf{a}_2(r,\pi/2)) + 4|\mathbf{p}_2(r,\pi/2) - \mathbf{q}_3(r)| + 2\mathcal{H}^1(\mathbf{a}_3(r)); \\ \mathcal{L}^2(G_{r,s} \cap S) &= 2(1 - 2\mathcal{L}^2(R_2(r,\pi/2)) - \mathcal{L}^2(R_3(r)); \\ \mathcal{L}^2(G_{r,s} \sim S) &= 2(1 - \mathcal{L}^2(R_1(s,\pi/2)))\end{aligned}$$

so that, with the help of Maple, we have

$$\Phi_G(r,s) = \frac{s}{2}((\pi-4)rs + (\pi-12)r^2 + 16r - 4).$$

Let

$$\sigma_G(r) = \frac{(3\pi - 12)r^2 + 16r - 4}{(4-\pi)r} \quad \text{for } r \in (0,\infty).$$

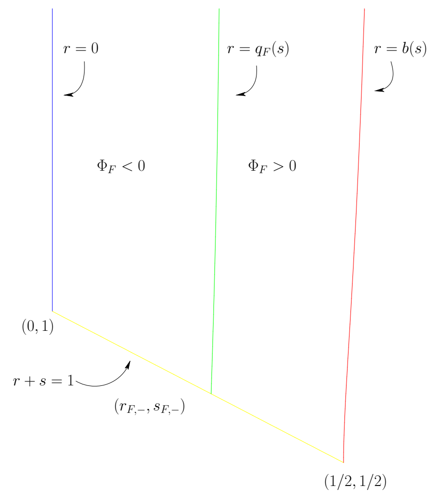


FIGURE 4. $W_F$ and the sets $\{\Phi_F < 0\}$ and $\{\Phi_F > 0\}$.

It follows that for any $(r,s) \in W_G$ we have

$$\begin{aligned}\Phi_G(r,s) < 0 &\Leftrightarrow s < \sigma_G(r);\\ \Phi_G(r,s) = 0 &\Leftrightarrow s = \sigma_G(r);\\ \Phi_G(r,s) > 0 &\Leftrightarrow s > \sigma_G(r).\end{aligned}$$

Let

$$r_{G,-} = 2\frac{4-\sqrt{4+3\pi}}{12-3\pi}, \quad s_{G,-} = 0,$$

note that $\{r : 0 < r < 1/2$ and $\sigma_G(r) = 0$ if and only if $r = r_{G,-}$ and let

$$r_{G,+} = r_{F,-}, \quad s_{G,+} = s_{F,-}.$$

Using Maple we find that $\sigma_G$ is increasing on $[r_{G,-}, r_{G,+}]$ from $s_{G,-} = 0$ to $s_{G,+}$ and that

$$r_{G,+} + s_{G,+} = 1.$$

It follows that

$$q_G = \{(\sigma_G(r), r) : 0 < r \leq r_{G,+}\}$$

is an increasing function with range equal $(0, s_{G,+}]$ such that
(19)
$\{(r,s) \in W_G : \Phi_H(r,s) < 0\} = \{(r,s) : s_{G,-} < s < s_{G,+}\ 0 < r < q_G(s) \text{ and } r+s \leq 1\}.$

7.4. **Analysis of $\Phi_H$.** Suppose $(r,s) \in W_H$. Then

$$\begin{aligned}\mathbf{TV}(H_{r,s}) &= 2\mathcal{H}^1(\mathbf{a}_1(s,\pi/2)) + 4|\mathbf{p}_1(s,\pi/2) - \mathbf{q}_2(r,\pi/2)|\\ &\quad + 4\mathcal{H}^1(\mathbf{a}_2(r,\pi/2)) + 4|\mathbf{p}_2(r,\pi/2) - \mathbf{q}_3(r)| + 2\mathcal{H}^1(\mathbf{a}_3(r));\\ \mathcal{L}^2(H_{r,s} \cap S) &= 2(1 - 2\mathcal{L}^2(R_2(r,\pi/2)) - \mathcal{L}^2(R_3(r));\\ \mathcal{L}^2(H_{r,s} \sim S) &= 2(1 - \mathcal{L}^2(R_1(s,\pi/2)))\end{aligned}$$

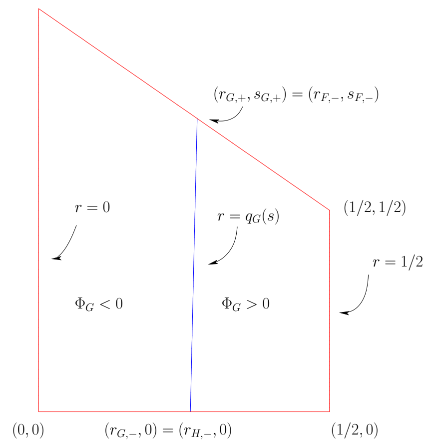


FIGURE 5. $W_G$ and the sets $\{\Phi_G < 0\}$ and $\{\Phi_G > 0\}$.

so that with the help of Maple we obtain

$$\begin{aligned} \Phi_H(r,s) &= 2s((\pi-4)r^2+4r-1) \\ &= -2s(4-\pi)(r-\rho_{H,-})(r-\rho_{H,+}) \end{aligned} \tag{20}$$

where

$$r_{H,-} = \frac{2-\sqrt{\pi}}{4-\pi} \approx 0.2650794522 \quad \text{and} \quad r_{H,+} = \frac{2+\sqrt{\pi}}{4-\pi} \approx 4.394712916.$$

In particular, $r_{H,-} < 1/2 < r_{H,+}$. It follows that for any $(r,s) \in W_H$ we have

$$\begin{aligned} \Phi_H(r,s) < 0 &\Leftrightarrow r < r_{H,-}; \\ \Phi_H(r,s) = 0 &\Leftrightarrow r = r_{H,-}; \\ \Phi_H(r,s) > 0 &\Leftrightarrow r > r_{H,-}. \end{aligned}$$

We also note that

$$(M_{r,s})_1(I_r) = (M_{r,s})_1(\alpha[I_r]) = 2(M_{r,s})_1(H_r).$$

In particular,

$$(M_{r,s})_1(H_r) < 0 \Rightarrow (M_{r,s})_1(H_r) < (M_{r,s})_1(I_r) = (M_{r,s})_1(\alpha[I_r]).$$

7.4.1. *Determination of* $\Psi$, $Q$, $q$ *and* $N$. Let $\Psi, Q, q, N$ be as in Section 5.

**Theorem 7.3.** We have

$$q = q_F \cup q_G \cup \{(r_{H,-}, s) : 0 < s < r_{H,-}\};$$

$$N = \{(q_F(s), s) : 0 < s < \infty\} \cup \{(r,r) : 0 < r < r_{H,-}\}.$$

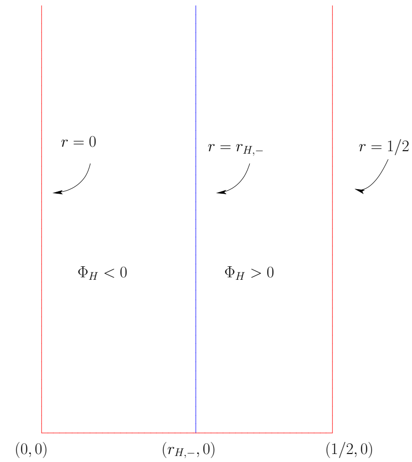


FIGURE 6. $W_H$ and the sets $\{\Phi_H < 0\}$ and $\{\Phi_H > 0\}$.

If $(r,s) \in \mathbb{P}$, $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\,[E]$ then

$$E \in \begin{cases} \{F_{r,s}\} & \text{if } r+s \geq 1,\ s \geq \sigma_{F,-} \text{ and } r < q_F(s), \\ \{F_{r,s}, \emptyset\} & \text{if } r+s \geq 1,\ s \geq \sigma_{F,-} \text{ and } r = q_F(s), \\ \{G_{r,s}\} & \text{if } r+s \leq 1,\ s > r \text{ and } r < q_G(s), \\ \{G_{r,s}, \emptyset\} & \text{if } r+s \leq 1,\ s > r \text{ and } r = q_G(s), \\ \{H_{r,s}\} & \text{if } r+s \leq 1,\ s < r \text{ and } r < r_{H,-}, \\ \{H_{r,s}, \emptyset\} & \text{if } r+s \leq 1,\ s < r \text{ and } r = r_{H,-}, \\ \{G_{r,s}, H_{r,s}\} & \text{if } r = s \text{ and } r < r_{H,-}, \\ \{G_{r,s}, H_{r,s}, I_{r,s}, J_{r,s}, \emptyset\} & \text{if } r = r_{H,-} \text{ and } s = r_{H,-}. \end{cases} \tag{21}$$

*Proof.* By a simple calculation one derives the following.

$$GH\Phi_G(r,s) - \Phi_H(r,s) = \frac{(4-\pi)rs(r-s)}{2} \quad \text{for } (r,s) \in W_G. \tag{22}$$

Keeping in mind the fact that if $(r,s) \in W_H$ and $(M_{r,s})_1(I_r) < 0$ then

$$(M_{r,s})_1(H_r) = 2\,(M_{r,s})_1(I_r) < (M_{r,s})_1(I_r)$$

we see that the last six of the eight cases in (21) hold.

Suppose $(r,s) \in Q \cap W_F$ and $r \leq 1/2$. I claim that $\Phi_F(r,s) < \Phi_H(r,s)$. Suppose, to the contrary, that $\Phi_H(r,s) \geq \Phi_F(r,s) \leq 0$, the last inequality a consequence of $(r,s) \in Q_F$. Since $\Phi_H(r,s) \leq 0$ we infer that $r \leq r_{H,-} < 1/2$. Choose $0 < s_3 < s_2 < s$ such that $r < s_2 < 1-r$ and $s_3 < r$. Thus $(r,s_2)$ is interior to $Q_G$ and $(r,s_3) \in Q_H$. By the preceding Proposition, $\Phi_G(r,s_2) < \Phi_H(r,s_2)$ so $G_{r,s_2} \in \mathbf{n}_1^{loc}(M_{r,s_2})$. We also have $H_r \in \mathbf{n}_1^{loc}(M_{r,s_3})$. But then Proposition 5.3 implies that $G_{r,s_2} \sim S \subset H_{r,s_3} \sim S$ which is obviously not the case. We may now infer that the first two cases of (21) hhold. (We could also have verified the claim using Maple.)

The formulae for $q$ and $N$ follow from (21) and our previous work. □

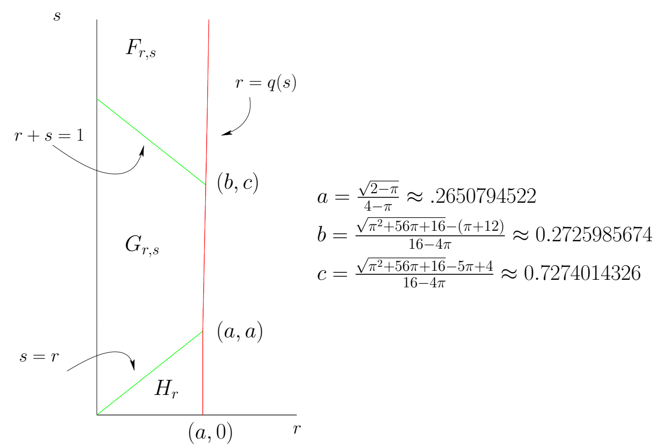


FIGURE 7. Illustration of Theorem 7.3.

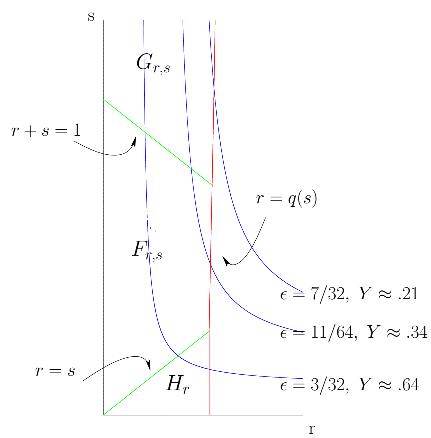


FIGURE 8. Minimizers for $\gamma(y) = y^2/2$ with $\epsilon = 7/32, 11/64, 3/32$; $Y$ here is the approximate maximum value of the minimizer.

7.5. **An analysis of three cases for $\gamma(y) = y^2/$, $y \in \mathbb{R}$.** We depict in the following figure the different possibilities when

$$F(f) = \frac{1}{2}\int |f(x) - 1_S(x)|^2 \, d\mathcal{L}^2 x, \quad \text{for } f \in \mathcal{F}(\mathbb{R}^2).$$

Notice the different character of the minimizer as $\epsilon$ varies.

## 8. Two circles.

Suppose $1 \le l < \infty$. Let $\mathbf{c}_\pm = (\pm l, 0)$, let $S_\pm = \mathbf{B}(\mathbf{c}_\pm, 1)$ and let

$$S = S_- \cup S_+.$$

We shall determine $\mathbf{n}_1^{loc}(M_{r,s})$, $(r, s) \in \mathbb{P}$.

$\Sigma = \{\iota, \alpha, \rho_1, \rho_2\}$ and note that $\Sigma$ is the group of rigid motions of $\mathbb{R}^2$ which carry $S$ into itself.

8.1. **Preliminaries.** For reasons which will become clear shortly we let

$$\begin{aligned}
G' &= (l-1,\infty) = \{s \in (0,\infty) : l < 1+s\};\\
\mathbf{d}_\pm(s) &= \pm\left(\arcsin\frac{l}{1+s}\right)\mathbf{e}_2 \quad \text{for } s \in G';\\
G &= \left(\frac{l^2-1}{2},\infty\right) = \{s \in (0,\infty) : l < \sqrt{2s+1}\};\\
\Theta(s) &= \arcsin\frac{l}{1+s} \quad \text{for } s \in G;\\
W_F &= \{(r,s) : 0 < r < \infty \text{ and } s \in G\};\\
L(t) &= \frac{1+\sqrt{1-t^2}}{t} \quad \text{for } t \in (0,1);\\
H &= \bigl(0, 2l/l^2+1\bigr) = \{t \in (0,1) : l < L(t)\};\\
X_F &= \{(r,t) : 0 < r < \infty \text{ and } t \in H\};\\
\zeta(t) &= \frac{l-t}{t} \quad \text{for } t \in H;\\
Z(r,t) &= (r,\zeta(t)) \quad \text{for } (r,t) \in X_F;\\
W_S &= \mathbb{P}
\end{aligned}$$

and we note that $Z$ carries $X_F$ diffeomorphically onto $W_F$.

The proof of the following Proposition is an elementary exercise which we leave to the reader.

**Proposition 8.1.** Suppose $0 < u < \infty$, $0 < s < \infty$ and $C$ is the circle with center $(0,u)$ and radius $s$. Then $C$ meets each of $S_+$ and $S_-$ tangentially if and only if $s \in G'$ and $(0,u) = \mathbf{c}_+(s)$ and $C \cap L_1 = \emptyset$ if and only if $s \in G$.

**Definition 8.1.** Suppose $s \in G$. We let

$$\begin{aligned}
A_-(s) &= \mathbf{a}(\mathbf{c}_-, s, \pi/2 - \Theta(s), \pi/2 + \Theta(s), s);\\
A_+(s) &= \rho_1[A_-(s)];\\
B_-(s) &= \mathbf{a}(\mathbf{d}_-(s), 1, \Theta(s), 2\pi - \Theta(s));\\
B_+(s) &= \rho_2[B_-(s)]
\end{aligned}$$

and we let

$$F_s$$

be the compact subset of $\mathbb{R}^2$ whose boundary is the union of the closures of the arcs $A_\pm(s), B_\pm(s)$.

8.2. **The main theorem.**

**Theorem 8.1.** Suppose $(r,s) \in \mathbb{P}$, $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\,[E]$. Then *either*

$$(r,s) \in W_F \text{ and } [E] \in \{0, [F_s]\}$$

*or*

$$(r,s) \in W_S \text{ and } [E] \in \{0, [S_-], [S_+], [S]\}.$$

**Remark 8.1.** More than one of these cases can occur.

The remainder of this section is devoted to the proof of this Theorem.

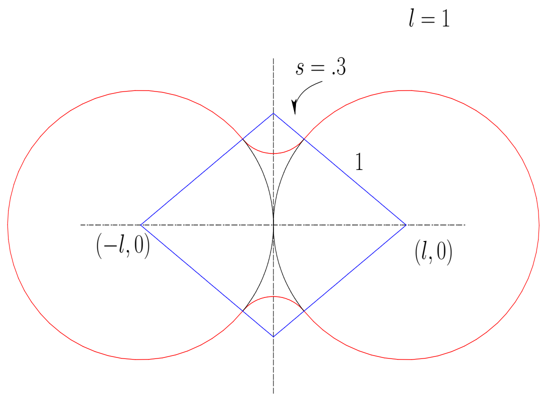


FIGURE 9. $F_s$ with $l = 1$ and $s = .3$.

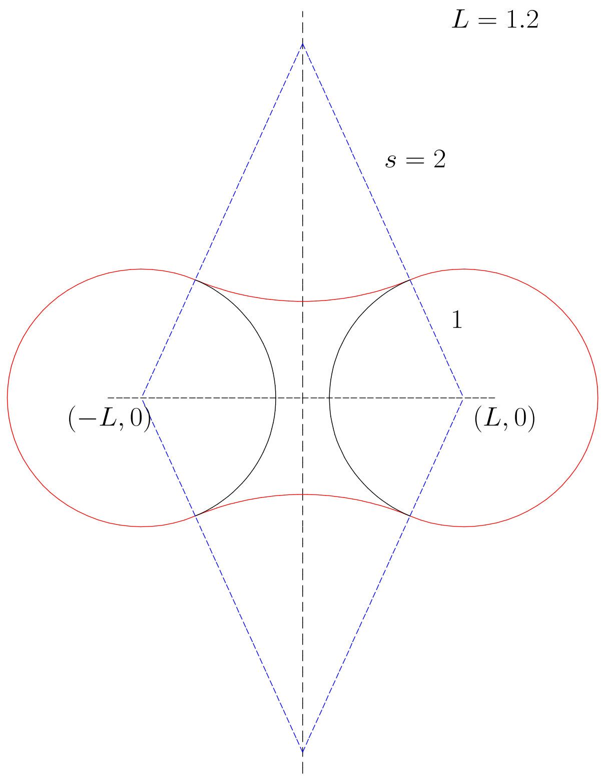


FIGURE 10. $F_s$ with $l = 1.2$ and $s = 2$.

8.2.1. We suppose throughout 8.2.1 that $0 < r < \infty$, $0 < s < \infty$ and

$$E \in \Gamma_{r,s}(S).$$

**Lemma 8.1.** Suppose $A \in \mathbf{ext}(E, S)$. Then $\mathbf{ends}(A)$ meets both $\mathbf{bdry}\, S_-$ and $\mathbf{bdry}\, S_+$.

*Proof.* Suppose the Lemma were false. Replacing $E$ by $\rho_2[E]$ if necessary we may assume that $\mathbf{ends}(A) \subset \mathbf{bdry}\, S_+$.

Let $m$ be the midpoint of $A$. Replacing $E$ by $\rho_1[E]$ if necessary we may assume that $m_2 \geq 0$. Let $a, b \in \mathbf{ends}(A)$ be such that $a_2 < b_2$. Then

$$b_2 = \frac{b_2 + a_2}{2} + \frac{b_2 - a_2}{2} > \frac{b_2 + a_2}{2} = m_2.$$

Let $Y = \mathbf{bdry}\, S_+$ and let $G$ be as in Theorem 6.1 (ii). Let $X$ be the connected component of $m$ in $G \cap \mathbf{bdry}\, E$, note that 6.1 (iv) holds, and let $P, Q$, et cetera, be as in Theorem 6.1. In particular, $N = \mathbf{card}\, Q < \infty$. From Proposition 6.1 we find that the circle containing $A$ meets $S_+$ transversely so $N \geq 2$. It follows from $(\Gamma 0)$ and $(\Gamma 1)$ that

$$Q \subset \{x \in \mathbb{R}^2 : x \bullet \mathbf{e}_1 < l\}. \tag{1}$$

so that there is $d \in Q$ such that

$$d \bullet \mathbf{e}_2 \geq x \bullet \mathbf{e}_2 \quad \text{for } x \in Q.$$

Thus there is $D \in \mathcal{R}_3$ such that $d \in \mathbf{ends}(D)$ and $\mathbf{ends}(D)$ meets $\mathbf{bdry}\, S_-$. But by Theorem 6.1 we have $D \subset \mathbf{B}(\mathbf{c}_+, |m - \mathbf{c}_+|)$ so we have arrived at a contradiction. □

**Lemma 8.2.** Suppose $E \subset S$. Then $E \in \{\emptyset, S_-, S_+, S\}$.

*Proof.* Were there $A \in \mathbf{int}(E, S)$ the circle containing $A$ would meet $\mathbf{bdry}\, S \sim \{0\}$ transversely by Proposition 6.1 which would force $E$ to meet the complement of $S$. □

8.2.2. *The main lemma.*

**Lemma 8.3.** Suppose $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\, [E]$. Then $\mathbf{ext}(E, S) \subset \{A_-(s), A_+(s)\}$.

**Remark 8.2.** By example one finds that the Lemma is false if we assume only that $E \in \Gamma_{r,s}(S)$.

*Proof.* Suppose $A \in \mathbf{ext}(E, S)$, $C$ is the circle containing $A$ and $c$ is its center.

By Lemma 8.1 there are $a_\pm \in \mathbf{bdry}\, S_\pm$ such that $\mathbf{ends}(A) = \{a_-, a_+\}$. This implies $A$ meets $L_2$ at a point $e$. By Theorem 6.2 $\mathbf{Tan}(\mathbf{bdry}\, E, e) = L_i$ for $i \in \{1, 2\}$. Were it the case that $\mathbf{Tan}(\mathbf{bdry}\, E), e) = L_2$ we would have that $\mathbf{ends}(A) \subset S_+$ or $\mathbf{ends}(A) \subset S_-$ which is incompatible with Lemma 8.1 so $\mathbf{Tan}(\mathbf{bdry}\, E, e) = L_1$. This implies that $c \in L_2$.

Suppose $C$ met $\mathbf{bdry}\, S \sim \{0\}$ transversely. Then there would be $B_\pm \in \mathbf{int}(E, S_\pm)$ such that $a_\pm \in \mathbf{ends}(B_\pm)$. Let $D_\pm$ be the circle containing $B_\pm$ and let $d_\pm$ be its center. By Proposition 6.2 we have

$$c,\ a_+,\ d_+ \quad \text{are collinear and} \quad c,\ a_-,\ d_- \quad \text{are collinear.} \tag{23}$$

Let $b_\pm$ be such that $\mathbf{ends}(B_\pm) = \{a_\pm, b_\pm\}$. By Proposition 6.1 the circle $D_\pm$ meets $S_\pm$ transversely at $b_\pm$. Were it the case that $b_- = b_+$ we would have $l = 1$ and $b_- = 0 = b_+$ which, by $(\Gamma 0)$, implies that $D_+ = D_-$ which is impossible given (23). So $b_- \neq b_+$. This implies there is $A' \in \mathbf{ext}(E, S)$ such that $b_+ \in \mathbf{ends}(A')$ Since $A, B_+, B_-$ are invariant under $\rho_1$ so is $A'$ and $\mathbf{ends}(A') = \{b_-, b_+\}$. Owing to (23) we find that *either* $A$ and $A'$ have length greater than $\pi s$ or $B_+$ and $B_-$ have lengths greater than $\pi r$. Thus $C$ meets $\mathbf{bdry}\, S$ transversely. Since the length of $A$ does not exceed $\pi s$ we conclude that $A = A_\pm(s)$. □

8.2.3. *Completion of the proof of the main theorem.* Suppose $(r,s) \in \mathbb{P}$, $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\,[E]$. In case $E \subset S$ we infer from Lemma 8.2 that $E \in \{\emptyset, S_-, S_+, S\}$ so suppose $E \sim S \neq \emptyset$ and choose $A \in \mathbf{ext}(E,S)$. By Lemma 8.3 we have $s \in G$ and $A = A_\pm(s)$. By $(\Gamma 0)$ we have $E \cap S_+ \neq \emptyset$ and $E \cap S_- \neq \emptyset$. Were it the case that $S \not\subset E$ there would be $B \in \mathbf{int}(E,S)$. The circle containing $A$ would meet either $\mathbf{bdry}\, S_+$ or $\mathbf{bdry}\, S_-$ transversely which is incompatible with Lemma 8.3. Thus $S \subset E$. We leave it to the reader to provide the straightforward arguement needed to show that $\rho_1[A] \in \mathbf{ext}(E,S)$ and that $E = F_s$.

8.3. **Calculations.** We let

$$\Phi_F(r,s) = rs\,(M_{r,s})_1\,(F_s) = rs\mathbf{TV}(F_s) - s\mathcal{L}^2(F_s \cap S) + r\mathcal{L}^2(F_s \sim S) \quad \text{for } (r,s) \in W_F,$$

$$\Phi_S(r,s) = rs\,(M_{r,s})_1\,(S) = rs\mathbf{TV}(S) - s\mathcal{L}^2(S) \quad \text{for } (r,s) \in W_S$$

and proceed to calculate these functions.

For $s \in H$ we have

$$(24)\quad \mathcal{H}^1(A_\pm(s)) = 2\Theta(s)s \quad \text{and} \quad \mathcal{H}^1(B_\pm(s)) = 2\pi - 2(\pi/2 - \Theta(s)) = \pi + 2\Theta(s).$$

**Proposition 8.2.** Suppose $s \in G$ and $\theta = \Theta(s)$. Then

$$(25)\quad \begin{aligned} \mathbf{TV}(F_s) &= 2\pi + 4\theta(1+s); \\ \mathcal{L}^2(F_s \cap S) &= 2\pi; \\ \mathcal{L}^2(F_s \sim S) &= 2l(1+s)\cos\theta + 2\theta(1-s^2) - \pi. \end{aligned}$$

*Proof.* The first of these equations follows immediately from (24) and the second is trivial.

For the third let $P$ be the parallelogram with vertices $\mathbf{c}_\pm$ and $\mathbf{d}_\pm(s)$; let $Q = \{(1-t)\mathbf{c}_+ + tx : x \in A_+(s)\}$; and let $R = \{(1-t)\mathbf{d}_+(s) + tx : x \in B_+(s)\}$. We have

$$\mathcal{L}^2(P) = 2l(1+s)\cos\theta), \quad \mathcal{L}^2(Q) = \frac{\pi - \theta}{2}, \quad \mathcal{L}^2(R) = \theta s^2$$

and

$$\mathcal{L}^2(F_s \sim S) = \mathcal{L}^2(P) - 2(\mathcal{L}^2(P) + \mathcal{L}^2(Q)).$$

$\square$

8.3.1. *Analysis of $\Phi_F$.* Let

$$\beta(t) = 2\arcsin(t) + 2t\sqrt{1-t^2}, \quad 0 < t < 1;$$

since $\beta'(t) = 4\sqrt{1-t^2} > 0$ for $0 < t < 1$ we infer that $\beta$ is increasing and $0 < \beta < \pi$.

For $(r,t) \in (0,\infty) \times (0,1)$ we let

$$\phi_F(r,t) = \frac{r\beta(t)l^2 + 2\pi t(r-1)l - \pi t^2(3r-2)}{t^2}.$$

Suppose $(r,s) \in W_F$ and $t = l/(1+s) = \arcsin\Theta(s)$. We use Maple to obtain

$$\Phi_F(r,s) = \phi_F(r,t).$$

With the help of Maple we find that

$$\beta(t)l^2 + 2\pi tl - 3\pi t^2 \geq \beta(t) + 2\pi t - 3\pi t^2 > 0 \quad \text{for } 0 < t < 1$$

so we may let

$$R(t) = \frac{2\pi t(l-t)}{\beta(t)l^2 + 2\pi tl - 3\pi t^2} \quad \text{for } 0 < t < 1$$

and deduce that

$$\phi_F(r,t) = 0 \ \Leftrightarrow \ r = R(t) \quad \text{for } (r,t) \in (0,\infty) \times (0,1).$$

With the help of Maple we find that

$$0 < R(t) < L(t) \quad \text{for } t \in (0,1). \tag{26}$$

Note that

$$\lim_{t \downarrow 0} R(t) = \frac{\pi/2}{\pi/2 + l} \quad \text{and that} \quad \lim_{t \uparrow 2l/(l^2+1)} R(t) = 1.$$

Let

$$q_F(s) = R(\Theta(s)) \quad \text{for } s \in G.$$

Then

$$\lim_{s \downarrow 2l/(l^2+1)} q_F(s) = 1 \quad \text{and} \quad \lim_{s \downarrow \infty} q_F(s) = \frac{\pi/2}{\pi/2 + l}.$$

Keeping in mind (26) we find that

$$\{(r,s) \in W_F : \Phi_F(r,s) < 0\} = \{(r,s) \in (0,\infty) \times G : r < q_F(s)\}. \tag{27}$$

As one can see using Maple it is *not* the case that $q_F$ is increasing.

8.3.2. *Analysis of* $\Phi_S$. For $(r,s) \in W_S$ we have

$$\Phi_S(r,s) = 2\pi s(2r-1)$$

so

$$\{(r,s) \in W_S : \Phi_F(r,s) < 0\} = \{(r,s) : 0 < r < 1/2 \text{ and } 0 < s < \infty\}. \tag{28}$$

8.3.3. *Relationship between* $\Phi_F$ *and* $\Phi_S$. Let

$$K(t,\lambda) = \beta(t)\lambda^2 - 2\pi t\lambda + \pi t^2 \quad \text{for } (t,\lambda) \in (0,1) \times (0,\infty).$$

With the help of Maple one finds that

$$\Phi_F(r,s) - \Phi_S(r,s) = \frac{r}{t^2} K(t,l). \quad \text{if } (r,s) \in W_F \text{ and } t = \arcsin \Theta(s) = l/(1+s).$$

Keeping in mind that $0 < \beta < \pi$ we let

$$\kappa_\pm(t) = \frac{t(\pi \pm \sqrt{\pi(\pi - \beta(t))})}{\beta(t)} \quad \text{for } 0 < t < 1.$$

Then

$$K(t,\lambda) = \beta(t)(\lambda - \kappa_+(t))(\lambda - \kappa_-(t)) \quad \text{for } (t,\lambda) \in (0,1) \times (0,\infty).$$

With the help of Maple one finds that

$$\kappa_- \text{ is increasing}, \quad \lim_{t \downarrow 0} \kappa_-(t) = 0 \quad \text{and} \quad \lim_{t \uparrow 1} \kappa_-(t) = 1$$

and

$$\kappa_+ \text{ is decreasing}, \quad \lim_{t \downarrow 0} \kappa_-(t) = \frac{\pi}{2} \quad \text{and} \quad \lim_{t \uparrow 1} \kappa_-(t) = 1.$$

One also finds with the help of Maple that

$$\kappa_+(t) < L(t) \quad \text{for } 0 < t < 1. \tag{29}$$

Let

$$\tau : (1, \pi/2) \to (0,1)$$

be the function inverse to $\kappa_+$ and note that $\tau$ is decreasing with range $(0,1)$.

So we have established the following Proposition.

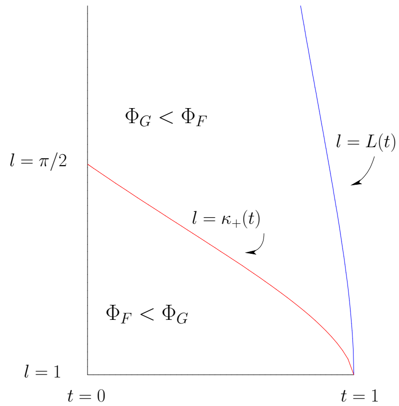


FIGURE 11. An illustration of Proposition 8.3

**Proposition 8.3.** Suppose $(r, s) \in W_F$ and $t = \arcsin \Theta(s) = l/(1+s)$. Then

$$\Phi_F(r,s) < \Phi_S(r,s) \;\Leftrightarrow\; t < \tau(l);$$
$$\Phi_F(r,s) = \Phi_S(r,s) \;\Leftrightarrow\; t = \tau(l);$$
$$\Phi_F(r,s) > \Phi_S(r,s) \;\Leftrightarrow\; t > \tau(l).$$

8.4. **Determination of $\Psi$, $Q$, $q$ and $N$.** Let $\Psi, Q, q, N$ be as in Section 5.

8.4.1. *Case One.* $l \ge \pi/2$.

**Theorem 8.2.** Suppose $l \ge \pi/2$, $(r, s) \in \mathbb{P}$, $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\,[E]$. Then

$$E \in \begin{cases} \{S\} & \text{if } r < 1/2, \\ \{S, S_+, S_-, \emptyset\} & \text{if } r = 1/2, \\ \{\emptyset\} & \text{if } r > 1/2, \end{cases}$$

*Proof.* Suppose $(r, s) \in W_F$ and let $t = \arcsin \Theta(s)$. Then

$$\Phi_F(r,s) - \Phi_S(r,s) = \frac{r}{t^2} K(t,l) > 0$$

since $l > \pi/2$ so $F_s \notin \mathbf{n}_1^{loc}(M_{r,s})$. The Theorem now follows from (28). □

8.4.2. *Case Two.* $1 < l < \pi/2$. Let

**Theorem 8.3.** Suppose $0 < l < \pi/2$ and $\sigma = \frac{l - \tau(l)}{\tau(l)}$. Then

$$Q = \{(s,r) : \sigma \le s < \infty \text{ and } s = q_F(r)\} \cup \{(s, 1/2) : 2l/(l^2+1) < s < \sigma\},$$
$$N = q^{-1} = \{(r,s) \in \mathbb{P} : r = q(s)\}$$

and, if $(r, s) \in \mathbb{P}$, $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\,[E]$, then

$$E \in \begin{cases} \{F_s\} & \text{if } \sigma < s < \infty \text{ and } 0 < r < q(s), \\ \{F_s, \emptyset\} & \text{if } \sigma < s < \infty \text{ and } r = q(s), \\ \{F_s, S, S_+, S_-, \emptyset\} & \text{if } r = 1/2 \text{ and } s = \sigma, \\ \{S\} & \text{if } 0 < r < 1/2 \text{ and } s < \sigma, \\ \{S, S_+, S_-, \emptyset\} & \text{if } r = 1/2 \text{ and } s < \sigma. \end{cases}$$

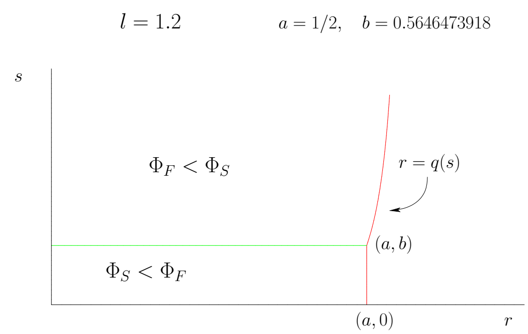


FIGURE 12. $Q$, $N$ and $q$ with separated circles.

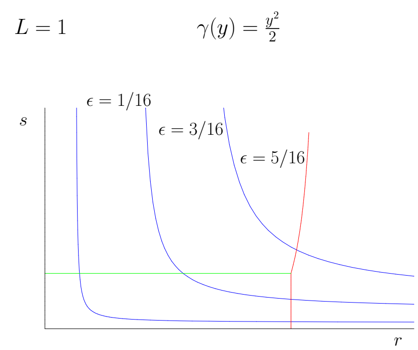


FIGURE 13. Minimizers for $\gamma(y) = y^2/2$ with $\epsilon = 1/16, 3/16, 5/16$; separated circles.

*Proof.* We make use of Proposition 8.3. □

8.4.3. *An analysis of three cases for* $\gamma(y) = y^2/$, $y \in \mathbb{R}$. We depict in the following figure the different possibilities when

$$F(f) = \frac{1}{2}\int |f(x) - 1_S(x)|^2 \, d\mathcal{L}^2 x, \quad \text{for } f \in \mathcal{F}(\mathbb{R}^2).$$

Notice the different character of the minimizer as $\epsilon$ varies.

8.4.4. *Case Three.* $l = 1$. Let

**Theorem 8.4.** Suppose $l = 1$. Then

$$q = \{(s, r) : 1 < s < \infty \text{ and } s = q_F(r)\},$$

$$N = q^{-1} = \{(r, s) \in \mathbb{P} : r = q(s)\}$$

and, if $(r, s) \in \mathbb{P}$, $E \in \mathbf{n}_1^{loc}(M_{r,s})$ and $E = \mathbf{spt}\,[E]$, then

$$E \in \begin{cases} \{F_s\} & \text{if } 1 < s < \infty \text{ and } 0 < r < q(s), \\ \{F_s, \emptyset\} & \text{if } 1 < s < \infty \text{ and } r = q(s). \end{cases}$$

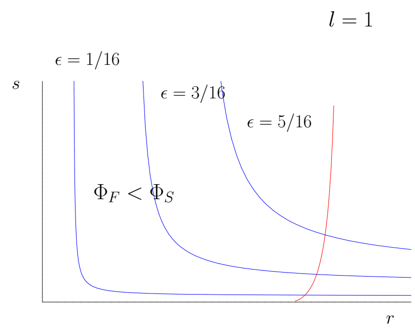


FIGURE 14. Minimizers for $\gamma(y) = y^2/2$ with $\epsilon = 1/16, 3/16, 5/16$; touching circles.

8.4.5. *An analysis of three cases for* $\gamma(y) = y^2/$, $y \in \mathbb{R}$. We depict in the following figure the different possibilities when

$$F(f) = \frac{1}{2} \int |f(x) - 1_S(x)|^2 \, d\mathcal{L}^2 x, \quad \text{for } f \in \mathcal{F}(\mathbb{R}^2).$$

Notice the different character of the minimizer as $\epsilon$ varies.

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