1. Preliminaries.

1.1. Tangents and normals.

Definition 1.1. Suppose $A \subset \mathbf{R}^n$ and a is an accumulation point of A. For each $r \in (0, \infty)$ we let

$$T_{a,r}(A) = \mathbf{cl}\,\{t(x-a) : 0 < t < \infty \text{ and } x \in \mathbf{B}(a,r) \sim \{a\}\}$$

and note that $T_{a,r}$ is a closed cone with vertex 0 in $\mathbf{R}^n$. We let

$$T_a(A) = \bigcap_{0<r<\infty} T_{a,r}(A).$$

We let

$$N_a(A) = \{v \in \mathbf{R}^n : v \bullet w \leq 0 \text{ whenever } v \in T_a(A)\}.$$

1.2. $g_{i,j}$ and $g^{i,j}$.

Exercise 1.1. Suppose V is an m-dimensional inner product space; v is a basic sequence for V; ω is the corresponding dual basic sequence for V^*; $\eta_i \in V^*$, $i = 1, \ldots, m$, is the image of v_i under the isomorphism from V to V^* induced by the inner product; and

$$g^{i,j}, \quad i,j \in \{1, \ldots, m\}$$

are such that

$$\sum_{k=1}^{m} g^{i,k} g_{k,j} = \delta^i_j$$

where we have set

$$g_{i,j} = v_i \bullet v_j \quad \text{for } i,j \in \{1,\ldots,m\} \quad \text{and} \quad \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Then

$$\omega_i = \sum_{k=1}^{m} g^{i,k} \eta_k \quad \text{for } i = 1, \ldots, m.$$

It follows that if $b : V \times V \to \mathbb{R}$ is bilinear and $L : V \to V$ is such that

$$L(v) \bullet w = b(v,w) \quad \text{for } v, w \in V$$

Then

$$\mathbf{trace}\, L = \sum_{i,j=1}^{n} g^{i,j} b_{i,j}. \tag{1}$$

As an important special case, suppose $w \in V$ and

$$g_{i,j} = \delta_{i,j} + (w \bullet v_i)(w \bullet v_j) \quad \text{for } i,j \in \{1, \ldots, m\}.$$

Then

$$g^{i,j} = \delta^{i,j} - \frac{(w \bullet v_i)(w \bullet v_j)}{1 + |w|^2}. \tag{2}$$

Indeed, if $L(v) = v + (v \bullet w)w$ for $v \in V$ one finds that

$$L^{-1}(v) = v - \frac{(v \bullet w)w}{1 + |w|^2} \quad \text{for } v \in V.$$

1.3. **Time slices.** Suppose X is a set. Whenever $A \subset X \times \mathbb{R}$ and $t \in \mathbb{R}$ we let
$$A_t = \{x : (x,t) \in A\}.$$
Whenever A is a subset of $X \times \mathbb{R}$, Y is a set and $f : A \to Y$ we let
$$f_t = \{(x, f(x,t)) : (x,t) \in A\};$$
note that
$$f_t : A_t \to Y.$$

2. Submanifolds.

Throughout this section we assume m and n are nonegative integers and $m \leq n$.

Definition 2.1. Suppose $M \subset \mathbf{R}^n$. We let
$$\mathcal{P}_m(M)$$
be the set of ordered pairs (U,P) such that

(i) U is a nonempty open subset of $\mathbf{R}^m$, $P : U \to M$, P is smooth and $\mathbf{rng}\, P \subset M$;

(ii) P is univalent and
$$\mathbf{dim}\,\mathbf{rng}\, \partial P(p) = m \quad \text{for each } p \in U;$$

(iii) for each $p \in U$ there is $r \in (0,\infty)$ such that $P^{-1}[\mathbf{B}(n,P(p))r$ is a compact subset of U.

Definition 2.2. We say a nonempty subset M of $\mathbf{R}^n$ is a **smooth m-dimensional submanifold of $\mathbf{R}^n$** if for each $a \in M$ there is $(U,P) \in \mathcal{P}_m(M)$ such that $a \in \mathbf{rng}\, P$. We let
$$\mathcal{M}_m(\mathbf{R}^n)$$
be the family of smooth m-dimensional submanifolds of $\mathbf{R}^n$.

Remark 2.1. If $M \in \mathcal{M}_m(\mathbf{R}^n)$, $(U,P) \in \mathcal{P}_m(M)$ and $p \in U$ then
$$T_{P(p)}(M) = \mathbf{rng}\, \partial P(p).$$

Theorem 2.1. Suppose M is a nonempty subset of $\mathbf{R}^n$. Then $M \in \mathcal{M}_m(\mathbf{R}^n)$ if and only if for each $a \in M$ there are an open neigborhood W of a and a smooth map $F : W \to \mathbf{R}^{n-m}$ such that

(i) $M \cap W = f^{-1}[\{f(a)\}]$;

(ii) $\mathbf{dim}\,\mathbf{rng}\, \partial F(x) = n - m$ for $x \in V$.

Remark 2.2. If M, a, W, f are as in the preceding definition we find that
$$T_a(M) = \mathbf{ker}\, \partial F(a).$$

Definition 2.3. Suppose $M \in \mathcal{M}_m(\mathbf{R}^n)$, Y is a finite dimensional vector space and $f : M \to Y$. We say f is **smooth** if $f \circ P$ is smooth whenever $(P,U) \in \mathcal{P}_m(M)$. For such an f and any $a \in M$ we let
$$\partial f$$
be the function with domain M which assigns to each point $a \in M$ the the linear map from $T_a(M)$ into Y characterized by the requirement that
$$\partial f(a) \circ \partial P(p) = \partial(f \circ P)(p)$$
whenever $(P,U,V) \in \mathcal{P}_m(\mathbf{R}^n)$, $p \in U$ and $a = P(p)$.

Definition 2.4. Suppose $M \in \mathcal{M}_m(\mathbf{R}^n)$. We let

$$\mathcal{X}(M)$$

be the vector space of smooth $X : M \to \mathbf{R}^n$ such that

$$X(x) \in T_x(M) \quad \text{whenever } x \in M.$$

We let

$$\mathcal{X}^\perp(M)$$

be the vector space of smooth $X : M \to \mathbf{R}^n$ such that

$$X(x) \in N_x(M) \quad \text{whenever } x \in M.$$

Definition 2.5. Suppose $M \in \mathcal{M}_m(\mathbf{R}^n)$ and $f : M \to \mathbb{R}$ is smooth. We let

$$\nabla f \in \mathcal{X}(M)$$

be such that

$$\nabla f(a) \bullet v = \partial f(a)(v) \quad \text{whenever } a \in M \text{ and } v \in T_a(M).$$

One calls ∇f the **gradient of f.**

3. Second fundamental forms and mean curvature.

Throughout this section we assume m and n are nonnegative integers, $m \leq n$ and $M \in \mathcal{M}_m(\mathbf{R}^n)$.

Let Π be the function with domain M whose value at $a \in M$ is orthogonal projection of $\mathbf{R}^n$ onto $T_a(M)$.

Exercise 3.1. Show the following:

(i) Π is smooth.

(ii) $\partial\Pi(a)(v)^* = \partial\Pi(a)(v)$ whenever $a \in M$ and $v \in T_a(M)$.

(iii)

$$\Pi(x) \circ \partial\Pi(x)(v) \circ \Pi(x) = 0 \quad \text{and} \quad \Pi(x)^\perp \circ \partial\Pi(x)(v) \circ \Pi(x)^\perp = 0$$

for $x \in M$ and $v \in T_x(M)$.

(iv)

$$\partial\Pi(x)(v)(w) = \partial\Pi(x)(w)(v) \quad \text{whenever } x \in M \text{ and } v, w \in T_x(M).$$

(Hint for the symmetry assertion: Whenever $(U, P) \in \mathcal{P}_m(M)$, $p \in U$ and $i \in \{1, \ldots, m\}$ we have

$$\Pi(P(p))(\partial_i P(p)) = \partial_i P(y) \quad \text{for } i \in \{1, \ldots, m\};$$

now apply ∂_j for $j \in \{1, \ldots, m\}$.)

Keeping in mind the preceding Exercise we define the **second fundamental form of M** to be the function

$$B$$

on M whose value at $a \in M$ is the symmetric bilinear map

$$B(a) : T_a(M) \times T_a(M) \to N_a(M)$$

defined by requiring that

$$B(a)(v, w) = \partial\Pi(a)(v)(w) \quad \text{for } a \in M \text{ and } v, w \in T_a(M).$$

It follows from the preceding Exercise that

$$B(a)(\partial_i P(p), \partial_i P(p)) = \Pi(a)^\perp(\partial_i \partial_j P(p)) \tag{3}$$

whenever $(U,P) \in \mathcal{P}_m(M)$, $p \in U$ and $a = P(p)$.

Thus the second fundamental form encodes the rate of change of the tangent space of M and depends on the inner product.

We also let

$$A$$

be the function on M whose value at $a \in M$ is a linear map from $N_a(M)$ into the symmetric linear maps from $T_a(M)$ to itself defined by requiring that

$$A(a)(u)(v) \bullet w = B(a)(v,w) \bullet u \quad \text{whenever } u \in N_a(M) \text{ and } v, w \in T_a(M).$$

It follows immediately that

$$(4) \qquad A(a)(u)(\partial_i P(b)) \bullet \partial_j P(b) = \partial_i \partial_j P(b) \bullet u \quad \text{whenever } u \in N_a(M)$$

whenever $(U,P) \in \mathcal{P}_m(M)$, $i, j \in \{1, \ldots, m\}$, $b \in U$ and $a = P(b)$.

Suppose N is differentiable normal vector field along M and $a \in M$. Since $\Pi(x) \circ N(x) = 0$ for $x \in M$ we find that, for any $v, w \in T_a(M)$,

$$\begin{aligned} \partial N(a)(v) \bullet w &= \Pi(a)(\partial N(a)(v)) \bullet w \\ &= -\partial \Pi(a)(v)(N(a)) \bullet w \\ &= -\partial \Pi(a)(v)(w) \bullet N(a) \\ &= -A(a)(N(a))(v) \bullet w; \end{aligned}$$

that is,

$$(5) \qquad \Pi(a) \circ \partial N(a) = -A(a)(N(a));$$

this is known as the *Weingarten formula*.

For example, if W is an open subset of $\mathbf{R}^n$, $f : W \to \mathbf{R}^{n-m}$ is smooth and $b \in \mathbf{R}^{n-m}$ is such that $M \cap W = \{f = b\}$ then, as $(\nabla f)|(M \cap W) \in \mathcal{X}^{\perp}(M \cap W)$ by virtue of (2.2), we find that

$$A(a)(\nabla f(a))(v) \bullet w = -\partial(\nabla f)(a)(v) \bullet w \quad \text{whenever } a \in M \text{ and } v, w \in T_a(M).$$

The **mean curvature vector of** M is, by definition, that member of $\mathcal{X}^{\perp}(M)$ such whose value at the point a of M satisfies

$$H(a) \bullet u = -\mathbf{trace}\, A(a)(u) \quad \text{whenever } u \in N_a(M);$$

in the classical literature the mean curvature vector is $1/m$ times H; hence the word "mean". It turns out the factor $1/m$ in is inconvenient when one is working, as we will be, with the first variation of area and for this reason we omit it. The direction of the mean curvature vector, and not just its magnitude, will be important in this work. It follows from (5) that

$$(6) \qquad H(a) \bullet Z(a) = \mathbf{trace}\, \Pi(a) \circ \partial Z(a)$$

whenever $Z \in \mathcal{X}^{\perp}(M)$ and $a \in M$.

For example, let $W = \mathbf{R}^n \sim \{0\}$, let $f(x) = |x|^2/2$ for $x \in W$, suppose $0 < R < \infty$ and let $M = \{x \in \mathbf{R}^n : |x| = R\}$. Then

$$\nabla f(x) = x$$

for $x \in W$. It follows from (5) that if $a \in M$ then

$$A(a)(a)(v) \bullet w = - \bullet w \quad \text{whenever } v, w \in T_a(M),$$

so

$$A(a) = -\Pi(a)$$

and

$$H(a) = \frac{n-1}{R^2} a.$$

Suppose $(U, P) \in \mathcal{P}_m(M)$. For each $y \in U$ and $i, j \in \{1, \ldots, m\}$ let

$$g_{i,j}(y) = \partial_i P(y) \bullet \partial_j P(y),$$
$$b_{i,j}(y) = \Pi(P(b))(\partial_i \partial_j P(y))$$

for each $y \in U$ and let

$$g^{i,j}(y)$$

be such that

$$\sum_{k=1}^{m} g^{i,k} g_{k,j} = \delta^i_j.$$

Then

$$H(P(y)) = \sum_{i,j=1}^{m} g^{i,j}(y) b_{i,j}(y) \quad \text{for } y \in U. \tag{7}$$

Finally, suppose $m = 1$ and $(U, P) \in \mathcal{P}_1(M)$. Then

$$g^{1,1}(t) = |P'(t)|^{-1} \quad \text{for } t \in U.$$

It follows that

$$\begin{aligned} H(P(t)) &= -|P'(t)|^{-2} \Pi(P(t))^{\perp}(P''(t)) \\ &= -|P'(t)|^{-4} \left(P''(t) \bullet P'(t)\, P'(t) - P''(t)\right) \end{aligned}$$

for $t \in U$.

Remark 3.1. It is clear we may replace $\mathbf{R}^n$ with any finite dimensional inner product space in the foregoing.

3.1. **Hypergraphs.** Perhaps the most important special case of the foregoing is the following. Suppose U is an open subset of $\mathbf{R}^m$ and $f : U \to \mathbb{R}$ is smooth. Let $n = m + 1$ and identify $\mathbf{R}^m \times \mathbb{R}$ with $\mathbf{R}^n$ in the usual way. Let

$$M = f = \{(x, y) \in \mathbf{R}^m \times \mathbb{R}) : y = f(x)\}.$$

Let

$$P(x) = (x, f(x)) \in \mathbf{R}^m \times \mathbb{R} \quad \text{for } x \in U.$$

Evidently, $(U, P) \in \mathcal{P}_m(M)$ and

$$\partial_i P = (\mathbf{e}_i, \partial_i f) \quad \text{and} \quad \partial_i \partial_j P = (0, \partial_i \partial_j f) \quad \text{for } i, j \in \{1, \ldots, m\}.$$

Let

$$g_{i,j}(x) = \partial_i P(x) \bullet \partial_j P(x) = (\mathbf{e}_i, \partial_i f(x)) \bullet (\mathbf{e}_j, \partial_j f(x)) = \delta_{i,j} + \partial_i f(x)\, \partial_j f(x)$$

for $x \in U$ and let

$$W = \sqrt{1 + |\nabla f|^2}.$$

Let

$$N(x, y) = W(x, y)^{-1}(-\nabla f(x), 1) \quad \text{for } x \in U$$

and note that $N|M$ is a unit normal field. We have from (2) that

$$g^{i,j} = \delta^{i,j} - \frac{\partial_i f\, \partial_j f}{W^2}$$

and

$$b_{i,j} = (\partial_i \partial_j P \bullet N)N = \frac{\partial_i \partial_j f}{W^2}(-\nabla f, 1)$$

so, by (7),

$$H \circ P = -\sum_{i,j=1}^{m} \left(\delta^{i,j} - \frac{\partial_i f \, \partial_j f}{W^2} \right) \frac{\partial_i \partial_j f}{W^2}(-\nabla f, 1). \tag{8}$$

4. Divergence and Laplacian.

Suppose m, n are nonnegative integers, $m \le n$ and $M \in \mathcal{M}_m(\mathbf{R}^n)$.

Definition 4.1. Suppose $X \in \mathcal{X}(M)$. We let

$$\mathbf{div}\, X,$$

the **divergence of** x, be the smooth function on M whose value at $a \in M$ equals

$$\mathbf{trace}\, \Pi(a) \circ \partial X(a).$$

The reason we introduce the divergence is because of the following Theorem, which will be proved in the next Section.

Theorem 4.1. Suppose $f \in \mathcal{E}(M)$, $X \in \mathcal{X}(M)$ and $\mathbf{spt}\, f \cap \mathbf{spt}\, X$ is compact. Then

$$\int_M Xf \, d\mathcal{H}^m = -\int_M f \, \mathbf{div}\, X \, \mathcal{H}^m.$$

Definition 4.2. Suppose Y is a finite dimensional vector space and $f : M \to Y$ is smooth. We let

$$\Delta f \in \mathcal{E}(M)$$

be such that

$$\omega(\Delta f) = \mathbf{div}\, \nabla(\omega \circ f) \quad \text{whenever } \omega \in V^*.$$

One calls Δf the **Laplacian** of f.

Theorem 4.2. Suppose Ω is an open set containing M, $f \in \mathcal{E}(\Omega)$ and $g = f|M$. Then

$$\Delta f(a) = \Delta g(a) + \nabla f(a) \bullet H(a) + \mathbf{trace}\, \Pi(a)^\perp \circ \partial(\nabla f)(a) \circ \Pi(a)^\perp \quad \text{for } a \in M.$$

Proof. Let $u_1, \ldots, u_n$ be an orthonormal basis for $\mathbf{R}^n$ such that $u_1, \ldots, u_m$ is an orthonormal basis for $T_a(M)$ and let $Z(x) = \nabla f(x) - \nabla g(x)$ for $x \in M$. Keeping in mind that $\nabla g(x) = \Pi(x)(\nabla f(x))$ for $x \in M$ we use the Weingarten equation to obtain

$$\begin{aligned}
\Delta g(a) &= \sum_{i=1}^{m} \partial(\nabla g)(a)(u_i) \bullet u_i \\
&= \sum_{i=1}^{m} \partial \Pi(a)(u_i)(\nabla f(a)) \bullet u_i + \Pi(a)(\partial(\nabla f)(a)(u_i)) \bullet u_i \\
&= \sum_{i=1}^{m} \partial \Pi(a)(u_i)(Z(x)) \bullet u_i + \partial(\nabla f)(a)(u_i) \bullet u_i \\
&= -H(x) \bullet Z(x) + \Delta f(x) - \sum_{i=m+1}^{n} \partial(\nabla f)(a)(u_i) \bullet u_i
\end{aligned}$$

□

Corollary 4.1. Suppose $X(x) = x$ for $x \in M$. Then

$$\Delta X = H.$$

Proof. For each $i = 1, \ldots, n$ let $Y_i(x) = x \bullet \mathbf{e}_i$ for $x \in \mathbf{R}^n$ and let $X_i = Y_i | M$. Then for each $i = 1, \ldots, n$ we have

$$(\Delta X) \bullet \mathbf{e}_i = \Delta X_i = \nabla Y_i \bullet H = H \bullet \mathbf{e}_i$$

since $\nabla Y_i = \mathbf{e}_i$. □

5. Jacobians and deformations.

Suppose m, n are nonnegative integers, $m \le n$ and $M \in \mathcal{M}_m(\mathbf{R}^n)$.

Definition 5.1. Jacobians. Suppose N is a positive integer and $F : M \to \mathbf{R}^N$ is smooth. We let

$$J_m F : M \to [0, \infty),$$

the **(m-dimensional) Jacobian of** F, be such that

$$J_m F(a) = \left| \bigwedge\nolimits_m \partial F(a) \right| = \sqrt{\mathbf{trace}\, \partial F(a)^* \circ \partial F(a)} \quad \text{whenever } a \in M.$$

We have

$$\int_M J_m F \, d\mathcal{H}^m = \int_{\mathbf{R}^N} N(F, y) \, d\mathcal{H}^m y \tag{9}$$

where for $y \in \mathbf{R}^N$ we have set $N(F, y)$ equal to the cardinality of $F^{-1}[\{y\}]$.

Suppose I is an interval, $0 \in I$ and

$$h : M \times I \to \mathbf{R}^n \text{ is smooth and } h(x, 0) = x \quad \text{for } x \in M.$$

Let $X = \dot{h}_0$ and let Y and Z be the tangential and normal components of X, respectively.

By the equality of mixed partials,

$$\frac{d}{dt} \partial h_t(x) \Big|_{t=0} = \partial X(x) = \partial Y(x) + \partial Z(x) \quad \text{for } x \in M.$$

Suppose $x \in M$. Let $u_1, \ldots, u_n$ be an orthonormal basis for $\mathbf{R}^n$ such that $u_i \in T_x(M)$ for $i = 1, \ldots, m$. For each $t \in I$ let

$$\xi(t) = \frac{d}{dt} \bigwedge\nolimits_{n-1} \partial h_t(x)(u_1 \wedge \cdots \wedge u_m).$$

$$\begin{aligned} \frac{d}{dt} \xi_t(x) \Big|_{t=0} &= \sum_{i=1}^m u_1 \wedge \cdots \wedge \partial X(x)(u_i) \wedge \cdots \wedge u_m \\ &= \left(\sum_{i=1}^m \partial X(x)(u_i) \bullet u_i \right) u_1 \wedge \cdots \wedge u_m \\ &\quad + \sum_{i=1}^m \left(\sum_{j=m+1}^n (\partial X(x)(u_i) \bullet u_j) u_1 \wedge \cdots \wedge u_j \wedge \cdots \wedge u_m \right) \end{aligned} \tag{10}$$

Keeping in mind that $|\xi(0)| = 1$ and invoking (6) we find that

$$
\begin{aligned}
\frac{d}{dt}|\xi(t)|\Big|_{t=0} &= \frac{\dot{\xi}(0) \bullet \xi(0)}{|\xi(0)|} \\
&= \sum_{i=1}^{m} \partial X(x)(u_i) \bullet u_i \\
&= \sum_{i=1}^{m} \partial Y(x)(u_i) \bullet u_i + \sum_{i=1}^{m} \partial Z(x)(u_i) \bullet u_i) \\
&= \mathbf{div}\, Y(x) + Z(x) \bullet H(x).
\end{aligned} \tag{11}
$$

It follows from the area formula (9) that

$$
\frac{d}{dt}\mathcal{H}^m(h_t[K])\Big|_{t=0} = \int_K X \bullet H \, d\mathcal{H}^m \tag{12}
$$

for any compact subset K of M such that $\mathbf{spt}\, X \subset K$.

Exercise 5.1. Derive a formula for the second variation of area.

6. Time indexed families of manifolds.

Suppose m and n are nonnegative integers, $m \le n$ and M is a smooth $m+1$ dimensional submanifold of $\mathbf{R}^n \times \mathbb{R}$. Let

$$
T(x,t) = t \quad \text{for } (x,t) \in \mathbf{R}^n \times \mathbb{R}.
$$

Proposition 6.1. Suppose $(x,t) \in M$. The following are equivalent:

(i) $\nabla(T|M)(x,t) \ne (0,0)$;
(ii) $T_{(x,t)}M \not\subset \mathbf{R}^n \times \{0\}$;
(iii) $(0,1) \notin N_{(x,t)}(M)$;

We now assume that

$$
\nabla(T|M)(x,t) \ne (0,0) \quad \text{whenever } (x,t) \in M. \tag{13}
$$

It follows that if $t \in \mathbb{R}$ and $M_t \ne \emptyset$ then M_t is an m-dimensional submanifold of $\mathbf{R}^n \times \mathbb{R}$. We say M is a **regular time indexed family of m-dimensional submanifolds of $\mathbf{R}^n$.**

Let

$$
V = |\nabla(T|M)|^{-2}\nabla(T|M) \in \mathcal{X}(()M);
$$

one calls V the **velocity of** M. Indeed, if I is an interval and (x,τ) is an integral curve of V then

Suppose ϕ is the flow of V. Let $x \in M_t$. Then

$$
\begin{aligned}
\frac{d}{ds}T(\phi_s(x,t)) &= \nabla T(\phi_s(x,t)) \bullet \dot{\phi}_s(x,t) \\
&= \nabla T(\phi_s(x,t)) \bullet V(\phi_s(x,t)) \\
&\nabla(T|M)(\phi_s(x,t)) \bullet V(\phi_s(x,t)) \\
&= 1
\end{aligned}
$$

It follows that

$$
\phi_s[M_t \times \{t\}] \subset M_{s+t} \times \{s+t\} \quad \text{whenever } s,t \in \mathbb{R}.
$$

Also,

$$1 = \nabla(T|M) \bullet V = \nabla T \bullet V = (0,1) \bullet V;$$

it follows that there is a smooth map

$$Z : M \to \mathbf{R}^n$$

such that

$$V(x,t) = (Z(x,t),1) \quad \text{for } (x,t) \in M.$$

Note that

$$Z_t(x) \in N_x(M_t) \quad \text{for } (x,t) \in M.$$

One calls Z the **normal velocity of** M. Evidently,

$$(|Z_t(x)|^2 + 1)|\nabla(T|M)(x,t)|^2 = 1.$$

6.1. **Parameterizations.**

Definition 6.1. Suppose $M \subset \mathbf{R}^n \times \mathbb{R}$. We let

$$\mathcal{Q}_m(M)$$

be the family of ordered pairs (U,Q) such that U is an open subset of $\mathbf{R}^m \times \mathbb{R}$, $Q : U \to \mathbf{R}^n$ and

$$(U_t, Q_t) \in \mathcal{P}_m(M_t) \quad \text{whenever } t \in \mathbb{R} \text{ and } U_t \neq \emptyset.$$

Suppose $(U,Q) \in \mathcal{Q}_m(M)$. Let

$$P(p,t) = (Q(p,t),t) \quad \text{for } (p,t) \in U$$

and let

$$M = \mathbf{rng}\, P.$$

One easily verifies that

$$(U,P) \in \mathcal{M}_{m+1}(M)$$

and that

$$(U_t, P_t) \in \mathcal{P}_m(M_t \times \{t\}) \quad \text{whenever } t \in \mathbb{R} \text{ and } M_t \neq \emptyset.$$

Moreover,

$$\dot{P}_t(p) = (\dot{Q}_t(p), 1)$$

has a nonzero inner product with $(0,1)$ so M satisfies (13). For each $(p,t) \in U$ let

$$Y(p,t)$$

be orthogonal projection of $\dot{Q}_t(p)$ on $T_{Q_t(p)}(M_t)$. Note that

$$(Y_t(p),1) \text{ is the orthogonal projection of } (\dot{P}_t(p),1) \text{ on } T_{P(p,t)}(M).$$

It follows that

$$\nabla(T|M)(P(p,t)) = \frac{1}{|Y_t(p)|^2 + 1}(Y_t(p),1).$$

This in turn implies that

$$Z_t(Q_t(p)) = Y_t(p) \quad \text{whenever } t \in \mathbb{R} \text{ and } p \in U_t.$$

6.2. **Implicit representation.** Suppose $m = n-1$, W is an open neighborhood of M, $u : W \to \mathbb{R}$ is smooth, $\nabla u(x,t) \neq (0,0)$ whenever $(x,t) \in W$ and

$$M = \{(x,t) \in W : u(x,t) = 0\}.$$

Suppose $(x,t) \in M$. Since

$$(0,0) \neq \nabla u(x,t) = (\nabla u_t(x), \dot{u}_t(x)) \in N_{(x,t)}(M)$$

and since $(0,1) \notin N_{(x,t)}(M)$ we have

$$\nabla u_t(x) \neq 0.$$

We also have

$$0 = V(x,t) \bullet \nabla u(x,t) = Z_t(x) \bullet \nabla u_t(x) + \dot{u}_t(x)$$

which implies

$$Z_t(x) = \frac{Z_t(x) \bullet \nabla u_t(x)}{|\nabla u_t(x)|^2} \nabla u_t(x) = -\frac{\dot{u}_t(x)}{|\nabla u_t(x)|^2} \nabla u_t(x). \tag{14}$$

6.2.1. *An example.* Let

$$\rho(x) = |x| \quad \text{and let} \quad \eta(x) = |x|^{-1}x \quad \text{for } x \in \mathbf{R}^n \times \{0\}.$$

Note that

$$\nabla \rho = \eta.$$

Suppose I is an interval, $f : I \to (0,\infty)$ and f' never vanishes. Let $W = (\mathbf{R}^n \sim \{0\}) \times I$ and let

$$u(x,t) = f(t) - |x| \quad \text{for } (x,t) \in W.$$

Then

$$\dot{u}(x,t) = f'(t) \quad \text{for } (x,t) \in W$$

so

$$M = \{(x,t) \in W : u(x,t) = 0\}$$

is a regular time indexed family of hypersurfaces in $\mathbf{R}^n \sim \{0\}$ with normal velocity

$$Z_t(x) = -\frac{\dot{u}_t(x)}{|\nabla u_t(x)|^2} \nabla u_t(x) = -f'(t)\eta(x).$$

Moreover, the mean curvature H_t of M_t is

$$\frac{1}{f(t)}\eta(x).$$

Thus M moves with velocity its mean curvature if and only if

$$f' = -\frac{1}{f}$$

so that, for any $t_0 \in I$,

$$f(t) = f(t_0) - \frac{(t-t_0)^2}{2} \quad \text{whenever } t \in I.$$

Given $R \in (0,\infty)$ we can set

$$I = (0, \sqrt{2R}) \quad \text{and} \quad f(t) = R - \frac{t^2}{2} \quad \text{for } t \in I.$$

7. Speed functions.

We're now going to assume $n > 0$ and $m = n - 1$.

Definition 7.1. Let
$$\mathbf{A}(n)$$
be the set of (x, u, α) such that $x \in \mathbf{R}^n$, $u \in \mathbf{S}^{n-1}$ and α is a symmetric endomorphism of
$$\mathbf{h}(u) = \{v \in \mathbf{R}^n : u \bullet v = 0\}.$$
Note that $\mathbf{A}(n)$ is a submanifold of $\mathbf{R}^n \times \mathbf{L}(\mathbf{R}^n, \mathbf{R}^n)$ of dimension $n + (n-1) + (n-1)n/2$. We say S is a **speed function** if
$$S : \mathbf{A}(n) \to \mathbb{R}$$
and S is smooth and
$$S(x, -u, -\alpha) = -S(x, u, A) \quad \text{whenever } (x, u, \alpha) \in \mathbf{A}(n).$$
(Of course we could soup this up and consider hihgher codimensions than 1 and speed functions that depend on more that two derivatives; but we won't.)

Suppose S is a speed function and M is a regular family of hypersurfaces in $\mathbf{R}^n$. We say M_t **moves with speed** S if
$$Z(x, t) = S(\psi_t(x), N_t(x), A_t(x)(N_t(x)))N_t(x)$$
whenever W is an open subset of M, $(x, t) \in W$, $N : W \to \mathbf{S}^{n-1}$ is continuous, $N_x(M_t) = \mathbf{h}(N_t(x))$ and $A_t(x)$ is the second fundamental form of M_t at x.

Interesting examples are $S(x, u, \alpha) = 1$ and $S(x, u, \alpha) = \mathbf{trace}\,\alpha$.

8. The level set method (I think!)

Let S be a speed function and let
$$F : \mathbf{R}^n \times \mathbf{L}(\mathbf{R}^n, \mathbb{R}) \times \mathbf{Sym}(\mathbf{R}^n) \to \mathbb{R}$$
be such that
$$F(x, \partial u(x), \partial^2 u(x)) = S(x, N(x), \alpha(x)),\ x \in M,$$
whenever u is a smooth real valued function on some open subset of $\mathbf{R}^n$, ∇u never vanishes, $c \in \mathbb{R}$, $M = \{u = c\}$, $N = |\nabla u|^{-1}\nabla u$, A is the second fundamental form of M and
$$\alpha(x) = A(x)(N(x)) \quad \text{for } x \in M.$$
Note that
$$\alpha = -\partial N = -\frac{1}{|\nabla u|}\partial(\nabla u) + \frac{\partial(\nabla u) \bullet \nabla u}{|\nabla u|^3}\nabla u.$$
For example, if $S \equiv 1$ we can let
$$F \equiv 1$$
and if $S(x, u, A) = -\mathbf{trace}\,A$ we can let F be such that
$$F(x, \partial u(x), \partial^2 u(x)) = \frac{1}{|\nabla u(x)|}\left(\Delta u(x) - \sum_{i,j} \frac{\partial_i u(x)\partial_j(x)}{|\nabla u(x)|^2}\partial_i\partial_j u(x)\right).$$

Exercise 8.1. Show that, given S, there is one and only one F as above and characterize the family of those F so obtained.

Suppose

$$u$$

is a real valued function on some open subset of space-time $\mathbf{R}^n \times \mathbb{R}$ such that

$$\nabla u_t(x) \neq 0$$

and

$$\dot{u}_t(x) + F(x, \partial u_t(x), \partial^2 u_t(x))|\nabla u_t(x)| = 0 \tag{15}$$

for (x,t) in the domain of u. We might call (15) the **level set equation for the speed S.** I hasten to add that this is the level set equation *in this context.* There are other contexts in which there are other level set equations. But the idea (?) is the same. Suppose $c \in \mathbb{R}$ and let

$$M = \{(x,t) : u(x,t) = c\}$$

and note that M is a regular time indexed family of hypersurfaces in $\mathbf{R}^n$. which is oriented by

$$N_t(x) = |\nabla u_t(x)|^{-1} \nabla u_t(x).$$

From (14) we find that

$$Z_t(x) = -\frac{\dot{u}_t(x)}{|\nabla u_t(x)|^2} \nabla u_t(x) = -\frac{\dot{u}_t(x)}{|\nabla u_t(x)|} N_t(x)$$

so that, indeed, M moves with speed S.

Here is a derivation using an explicit representation. Suppose $(U,Q) \in \mathcal{Q}_{n-1}(M)$. Differentiating

$$u(Q(p,t),t) = c, \ \ (p,t) \in U,$$

with respect to t we find that

$$\begin{aligned} 0 &= \dot{u}_t(Q_t(p)) + \nabla u_t(Q_t(p)) \bullet \dot{Q}_t(p) \\ &= -F(Q_t(p), \partial u_t(Q_t(p)), \partial^2 u_t(Q_t(p)))|\nabla u_t(Q_t(p))| + \nabla u_t(Q_t(p)) \bullet \dot{Q}_t(p) \\ &= |\nabla u_t(Q_t(p))\big(- S(Q_t(p), N_t(Q_t(p)), A_t(Q_t(p))) + \dot{Q}_t(p) \bullet N_t(Q_t(p))\big). \end{aligned}$$

which is to say that

$$\dot{Q}_t(p) \bullet N_t(Q_t(p)) = S(Q_t(p), N_t(Q_t(p)), A_t(Q_t(p))).$$

That is, M moves with speed S.

In case $S(x,u,A) = -\textbf{trace}\, A$ Evans and Spruck obtain large time viscosity solutions of this equation. The level set equation in this case is parabolic.

In case $S \equiv 1$ the level set equation is hyperbolic. I don't see that level set method helps in this case. Indeed, solving the equation in this case amounts to solving an eikonal equation. The two problems, in my opinion, are of equivalent difficulty.

9. Tubular neighborhoods and distance functions.

We let

$$\mathbf{Sym}(\mathbf{R}^n) = \{A \in \mathbf{L}(\mathbf{R}^n) : A^* = A\}.$$

Let

$$\mathbf{1} \in \mathbf{Sym}(\mathbf{R}^n)$$

be the identity map of $\mathbf{R}^n$:

$$\mathbf{1}(x) = x \quad \text{for } x \in \mathbf{R}^n.$$

Suppose M is a smooth hypersurface in $\mathbf{R}^n$ with unit normal field N. Let

$$\alpha : M \to \mathbf{Sym}(\mathbf{R}^n)$$

be defined by letting

$$\alpha(x) = A(x)(N(x)) \circ \Pi(x) \quad \text{for } x \in M$$

where A is the second fundamental form of M. For use later we let

$$\mathbf{F}(z, A) = A \circ (1 - zA)^{-1} \quad \text{for } A \in \mathbf{Sym}(\mathbf{R}^n) \text{ such that } |z|||A|| < 1.$$

This function will play a key role in what follows.

Let

$$G(x, r) = x + rN(x) \quad \text{for } (x, r) \in M \times \mathbb{R}.$$

Suppose $R \in (0, \infty)$ is such that G carries $M \times (-R, R)$ diffeomorphically onto some open neighborhood W of M.

Exercise 9.1. Show that such an R exists and that

$$|r|\,||\alpha(x)|| < 1 \quad \text{whenever } (x, r) \in G.$$

Let

$$(\xi, \rho) = F^{-1} : W \to M \times (-R, R),$$

let

$$\nu = N \circ \xi : G \to \mathbf{S}^{n-1}$$

and let

$$J : G \to \mathbf{R}^n$$

be such that $J|\{G(x, r) : x \in M\}$ is the mean curvature vector of $\{G(x, r) : x \in M\}$ whenever $r \in (-R, R)$.

9.1. **A key calculation.** Suppose $(x, r) \in M \times (-R, R)$ and let $y = G(x, r)$. Let $u_1, \ldots, u_{n-1} \in \mathbf{S}^{n-1}$ and $\lambda_1, \ldots, \lambda_{n-1}$ be such that

$$\partial N(x)(u_i) = \lambda_i\, u_i, \quad i = 1, \ldots, n-1.$$

From the Weingarten equation we obtain

$$\partial G(x, r)(u_i, 0) = (1 - r\lambda_i)u_i, \quad i = 1, \ldots, n-1$$

and it is obvious that

$$\partial G(x, r)(0, 1) = N(x).$$

It immediately follows that

$$\partial(\xi, \rho)(u_i, 0) = \frac{1}{1 - r\lambda_i} u_i, \quad i = 1, \ldots, n-1,$$

and

$$\partial(\xi, \rho)(y)(N(x)) = (0, 1).$$

It follows that

$$\partial\xi(y)(u_i) = \frac{1}{1 - r\lambda_i} u_i \quad \text{and} \quad \partial\rho(y)(u_i) = 0, \quad i = 1, \ldots, n-1$$

as well as

$$\partial\xi(y)(N(x)) = 0 \quad \text{and} \quad \partial\rho(y)(N(x)) = 1.$$

It follows directly from the foregoing that

$$\nabla\rho(y) = \nu(y).$$

By the chain rule we have

$$\partial\nu(x)(u_i) = \frac{\lambda_i}{1 - r\lambda_i} u_i \tag{16}$$

and

$$\partial\nu(x)(N(x)) = 0. \tag{17}$$

9.2. **A useful identity.** Evidently, $\nu|\{\rho = r\}$ is a unit normal field along $\{\rho = r\}$ for any $r \in (-R, R)$. It follows that

$$J \bullet \nu = \mathbf{div}\, \nu = \Delta\rho.$$

By (16) and (17) we obtain

$$\mathbf{div}\, \nu(y) = \mathbf{F}(r, \alpha(x)) \quad \text{whenever } (x, r) \in M \times (-R, R) \text{ and } y = G(x, y).$$