The Inverse and Implicit Function Theorems.

0.1. Proposition. Suppose $X$ and $Y$ are normed vector spaces and $L$ is a linear isomorphism from $X$ onto $Y$. Then

$$\frac{1}{\|L^{-1}\|} = \inf\{|L(x)| : x \in X \text{ and } |x| = 1\}.$$  

0.2. Remark. In what follows $1/\infty = 0$ and $1/\infty = 0$.

Proof. Set $\beta = \inf\{|L(x)| : x \in X \text{ and } |x| = 1\}$.

For any $x \in X$ such that $|x| = 1$ we have

$$1 = |L^{-1}(L(x))| \leq \|L\|^{-1}\|L(x)\|,$$

which implies that $1/\|L^{-1}\| \leq \beta$.

For any $y \in Y$ we have that

$$|y| = |L(L^{-1}(y))| \geq \beta\|L^{-1}(y)\|$$

which implies that $\|L^{-1}\| \leq 1/\beta$. \qed

0.3. The Inverse Function Theorem. Suppose

(1) $X$ and $Y$ are Banach spaces, $a \in X$, $0 < R < \infty$, $B = \{x \in X : |x - a| \leq R\}$ and $f : B \to Y$.

(2) $L$ is a linear isomorphism from $X$ onto $Y$, $\|L\| < \infty$ and

$$p(x) = f(x) - [f(a) + L(x - a)] \quad \text{for } x \in B.$$

(3) $\alpha < \beta$

where $\alpha = \text{Lip}(p)$ and $\beta = \inf\{|L(x)| : x \in X \text{ and } |x| = 1\}$.

Then

(4) $f^{-1}$ is a function and $\text{Lip}(f^{-1}) \leq (\beta - \alpha)^{-1}$;

(5) $\{f(a) + L(h) : |h| \leq (1-\alpha/\beta)R\} \subset \text{rng}(f) \subset \{f(a) + L(h) : |h| \leq (1+\alpha/\beta)R\}$;

(6) if $f$ is differentiable at $a$ with differential $L$ then $f^{-1}$ is differentiable at $f(a)$ with differential $L^{-1}$.

0.4. Remark. It is true and nontrivial that, if $L$ is a linear isomorphism from the Banach space $X$ onto the Banach space $Y$, the boundedness of $L$ is equivalent to the boundedness of $L^{-1}$. Thus there is some redundancy in our hypotheses.

Proof. By virtue of the previous Proposition we have that $\|L^{-1}\| = 1/\beta$. Note also that $p(a) = 0$.

Suppose $x_1$ and $x_2$ are points of $B$. We have

$$L(x_2 - x_1) = L(x_2 - a) - L(x_1 - a) = f(x_2) - f(x_1) - (p(x_2) - p(x_2))$$

so

$$|x_1 - x_2| = |L^{-1}(L(x_2 - x_1))|$$
Furthermore, for any $\alpha > 0$ we have
\[
\begin{align*}
&\leq \beta^{-1}(|p(x_2) - p(x_1)| + |f(x_1) - f(x_1)|) \\
&\leq \beta^{-1}(\alpha|x_2 - x_1| + |f(x_2) - f(x_1)|)
\end{align*}
\]
which proves (4).

Next we suppose $y \in \{f(a) + L(h) : |h| \leq (1 - \alpha/\beta)R\}$ and let $C : B \to Y$ be such that
\[
C(x) = a + L^{-1}(y - f(a) - p(x)) \quad \text{whenever } x \in B.
\]
For any $x \in B$ we have
\[
|C(x) - a| = |L^{-1}(y - f(a)) + L^{-1}(p(a) - p(x))| \\
\leq |L^{-1}(y - f(a))| + |L^{-1}(p(a) - p(x))| \\
\leq (1 - \alpha/\beta)R + \alpha/\beta|x - a|
\]
so
\[
C[B] \subset B.
\]
Furthermore, for any $x_1$ and $x_2$ in $B$ we have
\[
|C(x_1) - C(x_2)| = |L^{-1}(p(x_1) - p(x_2))| \leq \alpha/\beta|x_1 - x_2|.
\]
Thus, by the Contraction Mapping Principle, $C$ has a unique fixed point $x$ in $B$. Now
\[
C(x) = x \Rightarrow x = a + L^{-1}(y - f(a) - p(x)) \Rightarrow L(x - a) = y - f(a) - p(x) \Rightarrow f(x) = y;
\]
thus the first inclusion in (5) is proved.

To prove the second inclusion in (5), we suppose $x \in B$, set $h = L^{-1}(f(x) - f(a))$ and note that $h = (x - a) + L^{-1}(p(x) - p(a))$. Thus
\[
|h| = |(x - a) + L^{-1}(p(x) - p(a))| \leq |x - a| + \frac{\alpha}{\beta}|x - a| = (1 + \frac{\alpha}{\beta})|x - a|.
\]

Finally, suppose $f$ is differentiable at $a$ with differential $L$ and let $\epsilon > 0$. Let $\epsilon_f = \beta\epsilon/\delta - \alpha$. Choose $\delta_f$ such that
\[
x \in B \text{ and } |x - a| \leq \delta_f \Rightarrow |f(x) - f(a) - L(x - a)| \leq \epsilon_f|x - a|.
\]
Let $\delta = \delta_f/(\beta - \alpha)$ and suppose $y \in \text{rng } f$ and $|y - f(a)| \leq \delta$. Set $x = f^{-1}(y)$. Then
\[
|x - a| \leq \text{Lip } f^{-1}|y - f(a)| \leq \frac{1}{\beta - \alpha}|y - f(a)| \leq \delta_f
\]
so
\[
|f^{-1}(y) - a - L^{-1}(y - f(a))| \\
= |L^{-1}(L(x - a) - f(x) - f(a))| \\
\leq \beta\epsilon_f|x - a| \\
\leq \beta\epsilon_f \frac{1}{\beta - \alpha}|y - f(a)| \\
= \epsilon|y - f(a)|;
\]
since we know from (5) that $f(a)$ is an interior point of $\text{rng}(f)$ we conclude that $f^{-1}$ is differentiable at $f(a)$ with differential $L^{-1}$. $\square$
0.5. Corollary. Suppose
(1) $X$ and $Y$ are Banach spaces;
(2) $A \subset X$, $f : A \to Y$ and $f$ is differentiable at each point of $A$;
(3) $a \in A$, $\partial f$ is continuous at $a$ and $\partial f(a)$ is a Banach space isomorphism from $X$ onto $Y$.

Then there is an open subset $U$ of $A$ such that $a \in U$,
(4) $f[U]$ is an open subset of $Y$;
(5) $f[U]$ is univalent;
(6) $(f[U]^{-1})$ is differentiable at each point of $f[U]$.

Proof. We let 
$$
\beta(x) = \inf \{ |\partial f(x)(v)| : v \in X \text{ and } |v| = 1 \}
$$
whenever $x \in A$.

We let 
$$
\alpha(x, r) = \sup \{ |\partial f(y) - \partial f(x)| : y \in A \cap B_r(x) \}
$$
whenever $x \in A$ and $0 < r < \infty$.

Then 
$$
\alpha(x, r) \leq 2 \alpha(a, |x| + r)
$$
whenever $x \in A$ and $0 < r < \infty$.

Indeed, if $x \in A$, $0 < r < \infty$ and $y \in B_r(x)$ we have 
$$
|\partial f(y) - \partial f(x)| \leq |\partial f(y) - \partial f(a)| + |\partial f(x) - \partial f(a)| \leq \alpha(a, |x| + r) + \alpha(a, |x|).
$$

Moreover, if $0 < r < \infty$, $x, y \in B_r(a)$, $v \in X$ and $|v| = 1$ then 
$$
|\partial f(y)(v)| \geq |\partial f(x)(v)| - |\partial f(y)(v) - \partial f(x)(v)|
\geq \beta(x) - |\partial f(y) - \partial f(x)|
\geq \beta(x) - |\partial f(y) - \partial f(a)| - |\partial f(x) - \partial f(a)|
\geq \beta(x) - 2 \alpha(a, r)
$$
which implies that 
$$
\beta(y) \geq \beta(x) - 2 \alpha(a, r);
$$
thus 
$$
|\beta(y) - \beta(x)| \leq 2 \alpha(a, r)
$$
whenever $0 < r < \infty$ and $x, y \in B_r(a)$.

Since $A$ is open we may choose $R \in (0, \infty)$ such that $B_{2R}(a) \subset A$. Next we choose $b$ such that 
$$
0 < b < \beta(a);
$$
this is possible by because $\partial f(a)$ is a Banach space isomorphism from $X$ onto $Y$. Since $\partial f(a)$ is continuous at $a$ we may use (7) and (8) to choose $R_1$ such that 
$$
0 < R_1 \leq R,
$$

(9) 
$$
\alpha(x, R_1) < b \leq \beta(x)
$$
whenever $x \in B_{R_1}(a)$.

To complete the proof of the Theorem we need only apply the Inverse Function Theorem with $a, R, f$ there replaced by $x, R_1, f|B_{R_1}(x)$ for each $x \in B_{R_1}(a)$. □