1. More on the function $\mathbf{F}$.

Let

$$\mathbf{Z} = \{(z, A) \in \mathbb{R} \times \mathbf{Sym}(\mathbf{R}^n) : A \text{ is invertible and } |z|\, ||A^{-1}|| < 1\}$$

and note that $\mathbf{Z}$ is open. Let

$$\mathbf{F}(z, A) = \mathbf{trace}\, A \circ (\mathbf{1} - z\, A)^{-1} \quad \text{for } (z, A) \in \mathbf{G}.$$

Recall that if

$$\mathbf{inv} : \mathbf{GL}(\mathbf{R}^n) \to \mathbf{GL}(\mathbf{R}^n)$$

is inversion then

$$\partial \mathbf{inv}(A)(B) = -A^{-1} \circ B \circ A^{-1} \quad \text{whenver } A \in \mathbf{GL}(\mathbf{R}^n) \text{ and } B \in \mathbf{gl}(\mathbf{R}^n).$$

We have

$$\begin{aligned} \partial \mathbf{F}(z, A)(1, 0) &= \mathbf{trace}\, A \circ (\mathbf{1} - z\, A)^{-1} \circ A \circ (\mathbf{1} - z\, A)^{-1} \\ &= \mathbf{trace}\, A^2 \circ (\mathbf{1} - z\, A)^{-2} \\ &= A^2 \bullet (\mathbf{1} - z\, A)^{-2} \\ &\quad \text{as well as} \\ \partial \mathbf{F}(z, A)(0, B) &= \mathbf{trace}\, B \circ (\mathbf{1} - z\, A)^{-1} \\ &\qquad + A \circ (\mathbf{1} - z\, A)^{-1} \circ (zB) \circ (\mathbf{1} - z\, A)^{-1} \\ &= \mathbf{trace}\, B \circ (\mathbf{1} - z\, A)^{-1} \\ &\qquad + B \circ (\mathbf{1} - z\, A)^{-1} \circ z\, A \circ (\mathbf{1} - z\, A)^{-1} \\ &= \mathbf{trace}\, B \circ (\mathbf{1} - z\, A)^{-1} \circ \left(\mathbf{1} + z\, A \circ (\mathbf{1} - z\, A)^{-1}\right) \\ &= \mathbf{trace}\, B \circ (\mathbf{1} - z\, A)^{-2} \\ &= B \bullet (1 - z\, A)^{-2}. \end{aligned} \tag{1}$$

2. Gory formulae, part one.

Suppose v is a smooth function on some open subset Ω of $\mathbf{R}^n$. Let

$$w(x, t) = \frac{|\nabla v_t(x)|^2}{2} \quad \text{for } (x, t) \in \Omega.$$

I claim that

$$\nabla w_t = \partial(\nabla v_t)(\nabla v_t). \tag{2}$$

Here the right hand side evaluated at $x \in \Omega$ equals

$$\partial(\nabla v_t)(x)((\nabla v_t)(x)).$$

Reasonable, huh? Also,

$$\partial(\nabla w_t) = \partial(\partial(\nabla v_t))(\nabla v_t) + \partial(\nabla v_t) \circ \partial(\nabla v_t). \tag{3}$$

This means that if $x \in \Omega$ and $h \in \mathbf{R}^n$ then

$$\partial(\nabla w_t)(x)(h) = \partial(\partial(\nabla v_t)(x))(\nabla v_t(x))(h) + \big(\partial(\nabla v_t)(x) \circ \partial(\nabla v_t)(x)\big)(h).$$

Indeed, suppose $x \in \Omega$ and $h \in \mathbf{R}^n$. Then

$$\begin{aligned}\nabla w_t(x) \bullet h &= \partial w_t(x)(h) \\ &= \partial(\nabla v_t)(x)(h) \bullet \nabla v_t(x) \\ &= \partial(\nabla v_t)(x)(\nabla v_t(x)) \bullet h;\end{aligned}$$

this verifies (2). To verify (3) we use (2) to obtain

$$\begin{aligned}\partial(\nabla w_t)(x)(h) &= \partial(\partial(\nabla v_t)(x)(h)(\nabla v_t(x)) + \partial(\nabla v_t)(x)(\partial(\nabla v_t)(x)(h)) \\ &= \partial(\partial(\nabla v_t))(x)((\nabla v_t)(x)(h) + \partial(\nabla v_t)(x)(\partial(\nabla v_t)(x)(h)).\end{aligned}$$

3. Gory formulae, part two.

Suppose Γ is an open subset of $\mathbf{R}^n \times \mathbb{R}$, $v : \Gamma \to \mathbb{R}$ is smooth and

$$\dot{v}_t = \mathbf{F}(v_t, \partial(\nabla v_t)).$$

Let $w : \Gamma \to \mathbb{R}$ be such that

$$w_t = \frac{|\nabla v_t|^2}{2}$$

and let

$$G(x,t) = \mathbf{F}(v_t(x), \partial(\nabla v_t)(x)) \quad \text{for } (x,t) \in \Gamma.$$

Then

$$\dot{w}_t = \nabla \dot{v}_t \bullet \nabla v_t = \nabla G_t \bullet \nabla v_t.$$

For any $j \in \{1, \dots, n\}$ we have

$$\begin{aligned}\partial_j G_t &= \partial F(v_t, \partial(\nabla v_t))(\partial_j v_t, \partial_j(\partial(\nabla v_t))) \\ &= \Big(\partial_j v_t\, (\partial(\nabla v_t))^2 + \partial_j(\partial(\nabla v_t))\Big) \bullet C_t\end{aligned}$$

where $C : \Gamma \to \mathbf{Sym}(\mathbf{R}^n)$ is such that

$$C_t = (\mathbf{1} - v_t\, \partial(\nabla v_t))^{-2}.$$

It follows from (3) and (2) that

$$\begin{aligned}\dot{w}_t &= \Big(|\nabla v_t|^2 (\partial(\nabla v_t))^2 + \partial(\partial(\nabla v_t))(\nabla v_t)\Big) \bullet C_t \\ &= \Big(|\nabla v_t|^2 (\partial(\nabla v_t))^2 + \big(\partial(\nabla w_t) - \partial(\nabla v_t)^2\big)\Big) \bullet C_t \\ &= C_t \bullet \partial(\nabla w_t) + (\partial(\nabla v_t)^2 \bullet C_t)(2w_t - 1).\end{aligned} \tag{4}$$

Thus, if $c : \Gamma \to \mathbb{R}$ is such that

$$c_t = C_t \bullet (\partial(\nabla v_t))^2$$

and if $z : \Gamma \to \mathbb{R}$ is such that

$$z_t = |\nabla v_t|^2 - 1 = 2w_t - 1$$

we have

$$\dot{z}_t = C_t \bullet \partial(\nabla z_t) + c_t\, z_t.$$