Mathematics 282. Fall 2006. Elliptic Partial Differential Equations.

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Contents

1 Some basic notations and conventions.

We fix once and for all a positive integer n. Unless we say otherwise, Ω will be an open subset of $\mathbb{R}^n$.

We let

$$\mathbb{N}; \quad \mathbb{P}; \quad \mathbb{Z}; \quad \mathbb{Q}; \quad \mathbb{R}; \quad \mathbb{C}$$

be the set of nonnegative integers; the set of positive integers; the ring of integers; the field of rational numbers; the field of real numbers; and the field of complex numbers, respectively.

Whenever $a \in \mathbb{R}^n$ and $0 < r < \infty$ we let

$$\mathbf{U}(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\} \quad \text{and} \quad \mathbf{B}(a,r) = \{x \in \mathbb{R}^n : |x-a| \leq r\}.$$

We let

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

We let

$$\mathbf{e}_1, \ldots, \mathbf{e}_n \quad \text{and} \quad \mathbf{e}^1, \ldots, \mathbf{e}^n$$

be the standard basis vectors and covectors for $\mathbb{R}^n$ and its dual space, respectively.

We let

$$\mathbf{int}, \quad \mathbf{cl}, \quad \text{and} \quad \mathbf{bdry}$$

stand for "interior", "closure" and "boundary", respectively.

Whenever $A \subset \mathbb{R}^n$ and a is an accumulation point of A we let

$$\mathbf{Tan}(A,a) = \bigcap_{0<r<\infty} \mathbf{cl}\,\{t(x-a) : 0 < t < \infty \text{ and } x \in A \cap (\mathbf{B}(a,r) \sim \{a\})\}$$

and we let

$$\mathbf{Nor}(A,a) = \bigcap_{w \in \mathbf{Tan}(A,a)} \{v \in \mathbb{R}^n : v \bullet w \leq 0\}.$$

We let

$$\mathcal{L}^n$$

be n dimensional Lebesgue measure on $\mathbb{R}^n$.

If V is a vector space, $v \in V$ and ψ belongs to the dual space of V we frequently write

$$< v, \psi > \quad \text{instead of} \quad \psi(v).$$

Whenever $E \subset \Omega$ we let 1_E, the **indicator function of** E, be the function on Ω which is 1 on E and 0 on $\Omega \sim E$.

If X is a topological space, Y is a vector space and $f : X \to Y$ we let

$$\mathbf{spt}\, f = \mathbf{cl}\, \{x \in \Omega : X(x) \neq 0\}$$

and call this closed subset of X the **support of** f. We have

$$\mathbf{spt}\, f = \mathbf{cl}\, \{f \neq 0\} = \mathbf{cl}\, (X \sim \{f = 0\}) = X \sim \mathbf{int}\, \{f = 0\}.$$

Whenever $y, z \in \mathbb{R}$ we let

$$y \vee z = \max\{y, z\}, \quad \text{we let} \quad y \wedge z = \min\{y, z\}$$

and we note that $y + z = y \vee z + y \wedge z$.

Whenever X, Y, Z are sets, $F : X \to Y$, F is univalent, $\mathbf{rng}\, F = Y$ and $f : X \to Z$ we let $Ff = f \circ F^{-1} : Y \to Z$. Note that, under appropriate hypotheses, $(G \circ F)f = G(Ff)$.

Whenever $a \in \mathbb{R}^n$ and $0 < r < \infty$ we let

$$\tau_a(x) = x + a; \quad \mu_r(x) = rx; \quad A(x) = -x \quad \text{for } x \in \mathbb{R}^n.$$

Thus if Z is a set and $f : \mathbb{R}^n \to Z$ then

$$\tau_a f(x) = f(x - a), \quad \mu_r f(x) = f(r^{-1}x), \quad Af(x) = f(-x)$$

whenever $x \in \mathbb{R}^n$.

1.1 Multiindices.

We let

$$\mathbf{M}(n) = \mathbb{N}^{\{1,\ldots,n\}}.$$

We call a member of $\mathbf{M}(n)$ a **multiindex**. If α is a multiindex we set

$$\mathbf{w}(\alpha) = \sum_{i=1}^{n} \alpha_i$$

and call this natural number its **weight**. For each $m \in \mathbb{N}$ we let

$$\mathbf{M}(m, n) = \{\alpha \in \mathbf{M}(n) : \mathbf{w}(\alpha) = m\}.$$

Note that

$$\mathbf{card}\,\mathbf{M}(m,n) = \binom{m+n-1}{n-1}.$$

If α and $\beta \in \mathbf{M}(n)$ we write

$$\alpha \le \beta \quad \text{if} \quad \alpha_i \le \beta_i,\ i = 1,\ldots,n.$$

Whenever $\alpha \in \mathbf{M}(n)$ we let

$$\alpha! = \prod_{i=1}^{n} \alpha_i!$$

and we let

$$\mathbf{e}^\alpha(x) = x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i} \quad \text{for } x \in \mathbb{R}^n.$$

We let

$$\binom{m}{\alpha} = \frac{m!}{\alpha!}.$$

If $m \in \mathbb{N}$ we let $\mathbf{F}(m,n) = \emptyset$ if $m = 0$ and we let

$$\mathbf{F}(m,n) = \{1,\ldots,n\}^{\{1,\ldots,m\}} \quad \text{if } m > 0.$$

We have the mapping

$$\mathbf{a} : \mathbf{F}(m,n) \to \mathbf{M}(m,n)$$

defined by setting $\mathbf{a}(\emptyset)$ equal to the zero multiindex if $m = 0$ and by setting

$$\mathbf{a}(f)_i = \mathbf{card}\,\{j \in \{1,\ldots,m\} : f(j) = i\} \quad \text{for } f \in \mathbf{M}(m,n)$$

if $m > 0$.

By a straightforward combinatorial argument one finds that

$$\mathbf{card}\,\{f \in \mathbf{M}(m,n) : \mathbf{a}(f) = \alpha\} = \binom{m}{\alpha} \quad \text{for any } \alpha \in \mathbf{M}(m,n). \tag{1}$$

2 Differentiability and smoothness.

2.1 A brief excursion into the land of multilinear algebra.

We suppose throughout this section that X and Y are normed vector spaces and m is a nonnegative integer.

Whenever $m > 0$ and μ is an m linear function on X^m with values in Y we let

$$||\mu|| = \sup\{|\mu(x_1,\ldots,x_m)| : x_i \in X \text{ and } |x_i| \le 1,\ i = 1,\ldots,m\};$$

here and in what follows $|\cdot|$ may denote the norm in X as well as the norm in Y. Note that $||\mu|| < \infty$ if X is finite dimensional. We say μ is **bounded** if $||\mu|| < \infty$.

If $m > 0$ we let

$$\bigotimes^m (X, Y)$$

be the set of bounded m linear functions on X^m with values in Y. Note that $\bigotimes^m(X, Y)$ is a linear subspace of Y^{X^m} and is therefore a vector space; note also that $\bigotimes^m(X, Y)$ is complete if Y is complete. In case $m = 0$ we let $\bigotimes^m(X, Y) = Y$ and we let $||\mu|| = |\mu|$ for $\mu \in Y$.

We let

$$\bigodot^m (X, Y)$$

equal Y in case $m = 0$ and, in case $m > 0$, we let it be the linear subspace of symmetric members of $\bigotimes^m(X, Y)$. We have a natural linear map

$$\mathbf{sym} : \bigotimes^m (X, Y) \to \bigodot^m (X, Y)$$

defined by setting

$$\mathbf{sym}(\mu)(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma \in \mathbf{S}_m} \mu(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

$\mu \in \bigotimes^m(X, Y)$ and $x_i \in X$, $i = 1, \ldots, m$; here $\mathbf{S}_m$ is the group of permutations of $\{1, \ldots, m\}$. Evidently, $\mathbf{sym}(\mu) = \mu$ if and only if $\mu \in \bigodot^m(X, Y)$.

2.2 Homogeneous polynomial functions.

We define the linear map

$$\mathbf{p}^m : \bigotimes^m (X, Y) \to Y^X$$

by letting $\mathbf{p}^m(y) = y$ if $m = 0$ and $y \in \bigotimes^m(X, Y) = Y$ and, if $m > 0$, letting

$$\mathbf{p}^m(\mu)(x) = \mu(\underbrace{x, \ldots, x}_{m \text{ times}}) \quad \text{whenever } x \in X.$$

We let

$$\mathbf{P}^m(X, Y)$$

be the range of $\mathbf{p}^m$ and call its members **homogeneous polynomial functions of degree m on X with values in Y.**

Definition 2.2.1. Suppose $m \in \mathbb{N}$, $\alpha \in \mathbf{M}(m, n)$ and $\mu \in \bigodot^m(\mathbb{R}^n, Y)$. If $m = 0$ we have $\alpha = 0$ and we let $\mu^\alpha = \mu \in Y$; in case $m > 0$ we let

$$\mu^\alpha = \mu(\underbrace{\mathbf{e}_1, \ldots, \mathbf{e}_1}_{\alpha_1 \text{ times}}, \underbrace{\mathbf{e}_2, \ldots, \mathbf{e}_2}_{\alpha_2 \text{ times}}, \ldots, \underbrace{\mathbf{e}_n, \ldots, \mathbf{e}_n}_{\alpha_n \text{ times}}).$$

Proposition 2.2.1. Suppose $\mu \in \bigodot^m(\mathbb{R}^n, Y)$. Then

$$\frac{1}{m!} \mathbf{p}^m(\mu)(x) = \sum_{\mathbf{w}(\alpha) = m} \frac{x^\alpha}{\alpha!} \mu^\alpha \quad \text{whenever } x \in \mathbb{R}^n.$$

Proof. Suppose $x \in \mathbb{R}^n$. Keeping in mind (1) we find that

$$\begin{aligned}
\mu(\underbrace{x, \ldots, x}_{m \text{ times}}) &= \sum_{f \in \{1,\ldots,n\}^{\{1,\ldots,m\}}} \left(\prod_{i=1}^{n} x_{f(j)} \right) \mu(\mathbf{e}_{f(1)}, \ldots, \mathbf{e}_{f(m)}) \\
&= \sum_{\alpha \in \mathbf{M}(m,n)} \mathbf{card}\, \{f \in \mathbf{F}(m,n) : \mathbf{a}(f) = \alpha\} x^\alpha \mu^\alpha \\
&= \sum_{\mathbf{w}(\alpha)=m} \binom{m}{\alpha} x^\alpha \mu^\alpha.
\end{aligned}$$

□

Proposition 2.2.2. $\mathbf{p}^m$ carries $\bigodot^m(X, Y)$ isomorphically onto $\mathbf{P}^m(X, Y)$.

Proof. The Proposition holds trivially if $m = 0$ or $m = 1$ so suppose $m > 1$.

Since $\mathbf{p}^m = \mathbf{p}^m \circ \mathbf{sym}$ we find that $\mathbf{p}^m$ carries $\bigodot^m(X, Y)$ onto $\mathbf{P}^m(X, Y)$.

Suppose $\mu \in \bigodot^m(X, Y)$ and $\mathbf{p}^m(\mu) = 0$. Let $x_i \in X$, $i = 1, \ldots, m$. For each $t \in \mathbb{R}^m$ let

$$w(t) = \sum_{j=1}^{m} t_j x_j \in X$$

and $q \in \bigodot^m(\mathbb{R}^m, Y)$ be such that

$$q(t_1, \ldots, t_m) = \mu(w(t_1), \ldots, w(t_m)) \quad \text{whenever } t_i \in \mathbb{R}^m,\ i = 1, \ldots, m.$$

Note that

$$q^\alpha = \mu(\underbrace{x_1, \ldots, x_1}_{\alpha_1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{\alpha_2 \text{ times}}, \ldots, \underbrace{x_m, \ldots, x_m}_{\alpha_m \text{ times}}) \quad \text{for any } \alpha \in \mathbf{M}(m, m).$$

Suppose $t \in \mathbb{R}^m$. Then

$$0 = \frac{1}{m!}\mathbf{p}^m(\mu)(t) = \mu(\mathbf{w}(t), \ldots, \mathbf{w}(t)) = \frac{1}{m!} q(t, \ldots, t) = \sum_{\alpha \in \mathbf{M}(m,m)} \frac{t^\alpha}{\alpha!} q^\alpha.$$

It follows find that $q^\alpha = 0$ for all $\alpha \in \mathbf{M}(m, m)$. In particular, if $\iota \in \mathbf{M}(m, m)$ is such that $\iota(i) = i$ for $i = 1, \ldots, m$ then

$$0 = q^\beta = \mu(x_1, \ldots, x_m)$$

so $\mu = 0$ as desired. □

Remark 2.2.1. Let $\mathbf{T}(m) = \{1, -1\}^{\{1,\ldots,m\}}$. Using the notation of the proof of the preceding Proposition one has the nifty *polarization formula*

$$\mu(x_1, \ldots, x_m) = \frac{1}{2^m m!} \sum_{t \in \{1,-1\}^{\{1,\ldots,m\}}} \left(\prod_{i=1}^{m} t_i \right) \mu(w(t), \ldots, w(t)).$$

Keeping in mind the preceding Proposition, we let $||\cdot||$ be the norm on $\mathbf{P}(X, Y)$ such that $||\mathbf{p}^m(\mu))|| = ||\mu||$ whenever $\mu \in \bigodot^m(X, Y)$.

2.3 Differentiability.

Whenever $A \subset X$ and $f : A \to Y$ we let

$$\partial f$$

be the set of $(a, L) \in \mathbf{int}\, A \times \bigotimes^1 (X, Y)$ such that L is bounded and

$$\lim_{x \to a} \frac{|f(x) - f(a) - L(x-a)|}{|x-a|} = 0.$$

Note that ∂f is a function whose domain is a subset of the interior of A and whose range is a subset of $\bigotimes^1 (X, Y)$. We say f is **differentiable at** $a \in A$ if $a \in \mathbf{dmn}\, \partial f$.

For each nonnegative integer m we define

$$\partial^{\{m\}} f$$

by induction on m as follows. We set $\partial^{\{0\}} f = f$ and $\partial^{\{1\}} = \partial f$. If $m > 1$ then $\partial^{\{m+1\}} f$ is the set of

$$(a, \mu) \in A \times \bigotimes^{m+1} (X, Y)$$

such that $\partial^{\{m\}}$ is differentiable at a and

$$\mu(v_1, v_2, \ldots v_{m+1}) = \partial(\partial^{\{m\}} f)(a)(v_1)(v_2, \ldots, v_m)$$

whenever $(v_1, v_2, \ldots, v_{m+1}) \in X^{m+1}$. We say f is m **times differentiable at** a if $a \in \mathbf{dmn}\, \partial^{\{m\}} f$. It is evident that if f is $m+1$ times differentiable at a then

$$\partial^{\{m+1\}} f(a)(v_1, v_2, \ldots v_{m+1}) = \partial(\partial^{\{m\}} f)(a)(v_1)(v_2, \ldots, v_m)$$

whenever $(v_1, v_2, \ldots, v_{m+1}) \in X^{m+1}$.

We say f is **smooth** if f is m times differentiable at each point of a for all nonnegative integers m. Note that if f is smooth A must be open. In the real world things are not always smooth; unfortunately, this can really messes up the theory.

Theorem 2.3.1. If m is an integer, $m \geq 2$ and f is m times differentiable at a then

$$\partial^{\{m\}} f(a) \in \bigodot^m (X, Y).$$

Proof. This is well known. Can you prove it? □

Accordingly, we let

$$\partial^m f = \{(a, \mathbf{p}^m(\partial^{\{m\}} f(a)) : a \in A \text{ and } f \text{ is } m \text{ times differentiable at } a\}.$$

Evidently, $\partial^m f$ is a function which we call the m**th differential of** f. It is clear that $a \in \mathbf{dmn}\, \partial^m f$ if and only if f is m times differentiable at a in which case, as one easily verifies by induction on m,

$$\left(\frac{d}{dt}\right)^m f(a + th)\Big|_{t=0} = \partial^m f(a)(h) \quad \text{whenever } h \in X. \tag{2}$$

2.4 Partial and directional derivatives.

For any $v \in X$ we let

$$\partial_v f$$

be the set of $(a, w) \in \mathbf{int}\, A \times Y$ such that

$$w = \lim_{t \to 0} \frac{1}{t}(f(a + tv) - f(a)).$$

Evidently, $\partial_v f$ is a function whose domain is a subset of A and whose range is a subset of Y. It is called the **partial** or **directional derivative of** f **(in the direction** v**)**.

If your advanced calculus book does not have the following statement or the equivalent thereof get rid of it.

Theorem 2.4.1. Suppose X is finite dimensional, $v_1, \ldots, v_n$ is a basis for X, $A \subset X$, $f : A \to Y$, U is an open subset of A, $a \in U$, $m \in \mathbb{P}$ and, whenever $\{w_1, \ldots, w_m\} \subset \{v_1, \ldots, v_n\}$, the repeated directional derivative $\partial_{w_1} \ldots \partial_{w_m} f$ is exists at each point of U and is continuous at a. Then f is m times differentiable at a.

Remark 2.4.1. The corresponding statement without the continuity hypotheses is false.

When $X = \mathbb{R}^n$ we write

$$\partial_j \quad \text{for} \quad \partial_{\mathbf{e}_j}, \quad j = 1, \ldots, n.$$

Whenever $\alpha \in \mathbf{M}(n)$ we let

$$\partial^\alpha = \prod_{i=1}^{n} \partial_i^{\alpha_i}.$$

This last expression makes sense only in a context where $[\partial_i, \partial_j] = 0$ If A is a subset of $\mathbb{R}^n$, $f : A \to Y$, $a \in A$ and f is m times differentiable at a we have

$$\frac{1}{m!}\partial^m f(a)(h) = \sum_{\alpha \in \mathbf{M}(m,n)} \frac{h^\alpha}{\alpha!}\partial^\alpha f(a) \quad \text{for } h \in \mathbb{R}^n.$$

Theorem 2.4.2 (Leibniz' Rule)**.** Suppose m is a nonnegative integer $f, g : A \to \mathbf{R}$ are smooth and $a \in A$. Then

$$\partial^m (fg)(a)(x) = \sum_{l=0}^{m} \partial^l \binom{m}{l} f(a)(x) \partial^{m-l} g(a)(x) \quad \text{whenever } x \in \mathbb{R}^n.$$

Moreover,

$$\partial^\alpha (fg)(a) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(a) \partial^{\alpha - \beta} g(a).$$

Proof. Suppose $x \in \mathbb{R}^n$ and $I = \{t \in \mathbb{R} : (a, tx) \in <\Omega>\}$. Let $a(t) = f(a+tx)$ and let $b(t) = g(a+tx)$ for $t \in I$. One shows by induction on m that

$$(ab)^{(m)}(0) = \sum_{l=0}^{m} \binom{m}{l} a^{(l)}(0) b^{(m-l)}(0).$$

□

2.5 Taylor's Theorem.

As we will see shortly, Taylor's Theorem with integral remainder is a basic building block of the theory of generalized functions.

Exercise 2.5.1. Suppose $m > 0$ and ψ is a Y-valued function which is m times continuously differentiable on some open interval containing $[0, 1]$. Show that

$$\psi(1) \;=\; \sum_{0 \le l < m} \frac{1}{l!} \psi^{(l)}(0) - \int_0^1 \psi^{(m)}(t)\, d\frac{(1-t)^m}{m!}.$$

Hint: Induct on m using integration by parts.

For the remainder of this subsection we assume Ω is an open subset of $\mathbb{R}^n$ and $f : A \to Y$ is smooth.

Definition 2.5.1. We let

$$<\Omega>$$

be the set of $(a, h) \in \Omega \times \mathbb{R}^n$ such that $\{a + th : t \in [0, 1]\} \subset \Omega$ and note that $<\Omega>$ is an open subset of $\Omega \times \mathbb{R}^n$ containing $\Omega \times \{0\}$. For each $m \in \mathbb{N}$ we let

$$f^{<m>} : <\Omega> \to \mathbf{P}^m(\mathbb{R}^n, Y)$$

be such that

$$f^{<m>}(a, h)(v) = -\int_0^1 \partial^m f(a + th)(v)\, d\frac{(1-t)^m}{m!}$$

whenever $(a, h) \in <\Omega>$ and $v \in \mathbb{R}^n$.

Exercise 2.5.2 (Taylor's Theorem in many variables with integral remainder). For any positive integer m we have

$$f(a+h) = \sum_{l<m} \frac{1}{l!} \partial^l f(a)(h) + f^{<m>}(a, h)(h) \quad \text{whenever } (a, h) \in <\Omega>.$$

Hint: Suppose $(a, h) \in <\Omega>$,let I be the set of $t \in \mathbb{R}$ such that $a + th \in <\Omega>$, note that I is an open interval containing $[0, 1]$ and set

$$\psi(t) = f(a + th) \quad \text{for } t \in I.$$

Note that, by the Chain Rule and induction on $l \in \mathbb{N}$,

$$\psi^{(l)}(t) = \partial^l f(a + th)(h) \quad \text{whenever } t \in I.$$

Now make use of Exercise 2.5.1

3 Spaces of smooth functions.

3.1 The definitions.

Suppose X is a finite dimensional vector space and Y is a normed vector space. Let Ω be an open subset of X. We let

$$\mathcal{E}(\Omega, Y)$$

be the vector space of smooth functions on Ω with values in Y. For each nonnegative integer m and each compact subset K of Ω we let

$$\sigma_{m,K}(\phi) = \sup\{||\partial^l \phi(x)|| : 0 \le l \le m \text{ and } x \in K\} \quad \text{whenever } \phi \in \mathcal{E}(\Omega, Y).$$

Note that $\sigma_{m,K}$ is a seminorm on $\mathcal{E}(\Omega, Y)$. For each compact subset K of Ω we let

$$\mathcal{D}_K(\Omega, Y) = \{\phi \in \mathcal{E}(\Omega) : \mathbf{spt}\,\phi \subset K\}.$$

Note that $\mathcal{D}_K(\Omega) = \{0\}$ if $\mathbf{int}\,K = \emptyset$. We let

$$\begin{aligned}\mathcal{D}(\Omega, Y) &= \cup\{\mathcal{D}_K(\Omega, Y) : K \text{ is a compact subset of } \Omega\} \\ &= \{\phi \in \mathcal{E}(\Omega) : \mathbf{spt}\,\phi \text{ is compact}\}.\end{aligned}$$

We let

$$\mathcal{X}(\Omega) = \mathcal{E}(\Omega, \mathbb{R}^n)$$

be the vector space of smooth vector fields on Ω.

In case $Y = \mathbb{R}$ or $Y = \mathbb{C}$ we will frequently write

$$\mathcal{E}(\Omega), \quad \mathcal{D}_K(\Omega), \quad \mathcal{D}(\Omega)$$

for $\mathcal{E}(\Omega, Y), \mathcal{D}_K(\Omega, Y), \mathcal{D}(\Omega, Y)$, respectively.

3.2 Construction of lots of smooth functions with compact support.

How do we know there are any smooth functions with compact support? We'll make lots of them. They will come in handy.

Exercise 3.2.1. Suppose I is an open interval, $a \in I$, $f : I \to \mathbb{R}$, f is differentiable at each point of $I \sim \{a\}$ and

$$\lim_{x \to a} f'(x) = L$$

for some $L \in \mathbb{R}$. Prove that there is $M \in \mathbb{R}$ such that

$$\lim_{x \to a} f(x) = M.$$

Prove that g is differentiable at a and $g'(a) = L$ where

$$g(x) = \begin{cases} f(x) & \text{if } x \in I \sim \{a\}, \\ M & \text{if } x = a. \end{cases}$$

Exercise 3.2.2. A very useful example. We define

$$\phi : \mathbb{R} \to \mathbb{R}$$

by requiring that

$$\phi(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Show that

$$\mathbf{dmn}\, \phi^{(m)} = \mathbb{R} \quad \text{for each } m \in \mathbb{N}.$$

I suggest you proceed as follows.

(i) Use the chain rule and other rules for differentiation to show that

$$\mathbb{R} \sim \{0\} \subset \mathbf{dmn}\, \phi^{(m)} \quad \text{for each } m \in \mathbb{N}.$$

(ii) Show by induction that there is for each $m \in \mathbb{N}$ a polynomial function $p_m : \mathbb{R} \to \mathbb{R}$ such that

$$\phi^{(m)}(x) = e^{-\frac{1}{x}} p_m(x) \quad \text{whenever } x > 0.$$

(iii) Show that

$$\lim_{x \downarrow 0} e^{-\frac{1}{x}} \frac{1}{x^N} = 0 \quad \text{whenever } N \in \mathbb{N}.$$

(iv) Use (ii) and (iii) to show that

$$\lim_{x \to 0} \phi^{(m)}(x) = 0$$

for any $m \in \mathbb{N}$.

(v) Use Exercise 3.2.1 above to show that $0 \in \mathbf{dmn}\, \phi^{(m)}$ and $\phi^{(m)}(0) = 0$ for any $m \in \mathbb{N}$.

Proposition 3.2.1. Suppose $-\infty < c < d < \infty$. There is $\psi \in \mathcal{E}(\mathbb{R})$ such that $0 \le \psi \le 1$, $(-\infty, c] \subset \{\psi = 0\}$ and $[d, \infty) \subset \{\psi = 1\}$.

Proof. Let ϕ be as in Exercise 3.2.2. Let $\psi_1(x) = \psi(x - c)\psi(d - x)$ for $x \in \mathbb{R}$; evidently, $\psi_1 \in \mathcal{E}(\mathbb{R})$ and $\mathbf{spt}\, \psi_1 \subset [c, d]$. Let $I = \int_{-\infty}^{\infty} \psi_1(x)\, dx$ and let $\psi_2 = I^{-1}\psi_1$. Evidently, $\psi_2 \in \mathcal{E}(\mathbb{R})$, $\mathbf{spt}\, \psi_2 \subset [c, d]$ and $\int_{-\infty}^{\infty} \psi_2(x)\, dx = 1$. Let

$$\psi(x) = \int_{-\infty}^{\infty} \psi_2(y)\, dy \quad \text{for } x \in \mathbb{R}.$$

□

Proposition 3.2.2. Suppose $0 < r < s < \infty$ and $a \in \mathbb{R}^n$. There is $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \le \psi \le 1$, $\mathbf{spt}\, \psi \subset \mathbf{U}(a, s)$ and $\mathbf{B}(a, r) \subset \mathbf{int}\, \{\psi = 1\}$.

Proof. Let c, d be real numbers such that $r < c < d < s$. Let ψ_1 be as in the preceding Proposition. Finally, let

$$\psi(x) = 1 - \psi_1(|x - a|) \quad \text{whenever } x \in \mathbb{R}^n.$$

□

Corollary 3.2.1. Suppose U is an open subset of $\mathbb{R}^n$ and K is a compact subset of U. There is $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \le \psi \le 1$, $K \subset \mathbf{int}\,\{\psi = 1\}$ and $\mathbf{spt}\,\psi \subset U$.

Proof. Let N be a positive integer such that there exist points $a_1, \ldots, a_m$ of K and radii $r_1, \ldots, r_N$ such that

$$K \subset \cup_{i=1}^{n} \mathbf{B}(a_i, r_i) \subset U.$$

Choose $s_1, \ldots, s_N$ such that, for each $i = 1, \ldots, N$, $r_i < s_i < \infty$ and $\mathbf{B}(a_i, s_i) \subset U$. Use the preceding Proposition to obtain $\psi_i \in \mathcal{D}(\mathbb{R}^n)$ such that, for each $i = 1, \ldots, N$, $0 \le \psi_i \le 1$, $\mathbf{spt}\,\psi_i \subset \mathbf{U}(a_i, s_i)$ and $\mathbf{B}(a_i, r_i) \subset \mathbf{int}\,\{\psi_i = 1\}$. Let

$$\psi = 1 - \prod_{i=1}^{N} (1 - \psi_i).$$

□

Exercise 3.2.3. Suppose c is a sequence in $\mathbb{C}$. Show that there exists $f \in \mathcal{D}(\mathbb{R})$ such that

$$f^{(m)}(0) = \frac{c_m}{m!} \quad \text{whenever } m \in \mathbb{N}.$$

Hint: Let $\psi \in \mathcal{D}(\mathbb{R})$ be such that $0 \le \psi \le 1$, $\mathbf{spt}\,\psi \subset [0, 1]$ and $0 \in \mathbf{int}\,\{\psi = 1\}$. Choose a sequence M of positive real numbers such that the series

$$f(x) = \sum_{m=0}^{\infty} \psi(M_m x) \frac{c_m}{m!} x^m, \quad x \in \mathbb{R},$$

together with all its derivative converges uniformly. It suffices to show that for each $N \in \mathbb{N}$ one has

$$\sum_{m=N}^{\infty} \sup \left\{ \left| \left(\frac{d}{dx} \right)^N \psi(M_m x) \frac{c_m}{m!} x^m \right| : x \in \mathbb{R} \right\} < \infty.$$

The $m!$'s are irrelevant.

Exercise 3.2.4. Suppose F is a closed subset of $\mathbb{R}^n$. Show that there is $f \in \mathcal{E}(\mathbb{R}^n)$ such that $f \ge 0$ and $F = \{f = 0\}$.

3.3 Partitions of unity.

Theorem 3.3.1. Suppose Ω is an open subset of $\mathbb{R}^n$ and $\mathcal{U}$ is an open covering of Ω. There is a subfamily Φ of $\mathcal{D}(\Omega)$ such that

(i) $\{\phi \in \Phi : K \cap \mathbf{spt}\,\phi \neq \emptyset\}$ is finite whenever K is a compact subset of Ω;

(ii) for each $\phi \in \Phi$ there is $U \in \mathcal{U}$ such that $\mathbf{spt}\,\phi \subset U$;

(iii) $\sum_{\phi\in\Phi} \phi(x) = 1$ whenever $x \in \Omega$.

Proof. Let $C_0 = \{x \in \mathbb{R}^n : 0 \le x_i \le 1,\ i = 1, \ldots, n\}$. For each $m \in \mathbb{N}$ let

$$\mathcal{C}_m = \{\mu_{2^{-m}}(z + C_0) : z \in \mathbb{Z}^n\}$$

and for each $C \in \mathcal{C}_m$ let

$$\hat{C} = \{x \in \mathbb{R}^n : \mathbf{dist}\,(x, C) \le 2^{-m}\}.$$

For each $m \in \mathbb{N}$ define subfamilies $\mathcal{D}_m$ of $\mathcal{C}_m$ as follows: $C \in \mathcal{D}_0$ if $C \in \mathcal{C}_0$ and $\hat{C} \subset U$ for some $U \in \mathcal{U}$; if $m \in \mathbb{N}$ then $C \in \mathcal{D}_{m+1}$ if $C \in \mathcal{C}_{m+1}$, $\hat{C} \subset U$ for some $U \in \mathcal{U}$ and C is not contained in any member of $\mathcal{D}_m$.

Let $\mathcal{D} = \cup_{m=0}^{\infty} \mathcal{D}_m$. Note that $\Omega = \cup\mathcal{D}$ and that

$$\{\hat{C} : C \in \mathcal{D} \text{ and } \hat{C} \cap K \neq \emptyset\}$$

is finite for any compact subset K of Ω. For each $C \in \mathcal{D}$ let $\psi_C \in \mathcal{D}(\Omega)$ be such that $0 \le \psi_C \le 1$, $C \subset \{\psi_C = 1\}$ and $\mathbf{spt}\,\psi_C \subset \hat{C}$. Let $\psi = \sum_{C\in\mathcal{D}} \psi_C$; note that $\psi \in \mathcal{E}(\Omega)$ and that $\psi(x) > 0$ whenever $x \in \Omega$. Let $\Phi = \{\frac{\psi_C}{\psi} : C \in \mathcal{D}\}$. □

4 Classical convolution.

Whenever a, b are nonnegative extended real valued measurable functions on $\mathbb{R}^n$ we define the nonnegative extended real valued measurable function

$$a * b$$

on $\mathbb{R}^n$ by letting

$$a * b(x) = \int a(x - y)b(y)\,dy \quad \text{for } x \in \mathbb{R}^n.$$

$a * b$ is called the **convolution of** a **and** b.

Suppose f, g are complex valued measurable functions on $\mathbb{R}^n$. Let

$$E(f, g)$$

be the set of $x \in \mathbb{R}^n$ such that

$$\int |f(x - y)g(y)|\,dy = |f| * |g|(x) = \infty$$

and note that $E(f,g)$ is measurable. If $\mathcal{L}^n(E(f,g)) = 0$ we define we define the complex valued measurable function

$$f * g$$

on $\mathbb{R}^n$ by letting

$$f * g(x) = \begin{cases} \int f(x-y)g(y)\,dy & \text{if } x \notin E(f,g), \\ 0 & \text{if } x \in E(f,g). \end{cases}$$

$f * g$ is called the **convolution of f and g**; evidently,

$$|f * g| \leq |f| * |g|.$$

We set

$$\frac{1}{\infty} = 0.$$

Whenever $1 \leq s \in [1,\infty]$ we let $s'[1,\infty]$ be such that

$$\frac{1}{s} + \frac{1}{s'} = 1.$$

One says s and s' are **conjugate**.

Theorem 4.0.2. Suppose a, b, c are nonnegative extended real valued measurable functions on $\mathbb{R}^n$. Then

$$a * b = b * a$$

and

$$a * (b * c) = (a * b) * c.$$

Suppose f, g, h are complex valued measurable functions on $\mathbb{R}^n$. Then $E(f,g) = E(g,f)$ and if $\mathcal{L}^n(E(f,g)) = 0$ then

$$f * g = g * f.$$

Moreover,

$$\mathcal{L}^n(E(g,h)) = 0 \text{ and } \mathcal{L}^n(E(f, g * h)) = 0$$

$$\Leftrightarrow$$

$$\mathcal{L}^n(E(f,g)) = 0 \text{ and } \mathcal{L}^n(E(f * g, h)) = 0$$

in which case

$$f * (g * h)(x) = (f * g) * h(x) \quad \text{for almost all } x.$$

Exercise 4.0.1. Prove the preceding Theorem.

Theorem 4.0.3 (Young's Inequality)**.** Suppose $p, q \in [1, \infty]$,

$$\frac{1}{p} + \frac{1}{q} \geq 1.$$

and $r \in [1, \infty]$ is such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Then

$$||a * b||_r \leq ||a||_p ||b||_q$$

whenever a, b are nonnegative extended real valued measurable functions on $\mathbb{R}^n$.

Moreover, if $f \in \mathbf{L}_p(\mathbb{R}^n)$ and $g \in \mathbf{L}_q(\mathbb{R}^n)$ then $f * g$ is defined and

$$||f * g||_r \leq ||f||_p ||g||_q.$$

Proof. If $p = 1$ then $q = \infty$ and $r = \infty$ and if $q = 1$ then $p = \infty$ and $r = \infty$; in either of these cases the inequalities hold trivially, so we henceforth assume both p and q are finite and this implies r is finite.

We have

$$\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1 - \frac{1}{p} + 1 - \frac{1}{q} + \frac{1}{p} + \frac{1}{q} - 1 = 1,$$

$$1 - \frac{p}{r} = 1 - p\left(\frac{1}{p} + \frac{1}{q} - 1\right) = p\left(1 - \frac{1}{q}\right) = \frac{p}{q'}$$

and

$$1 - \frac{q}{r} = 1 - q\left(\frac{1}{p} + \frac{1}{q} - 1\right) = q\left(1 - \frac{1}{p}\right) = \frac{q}{p'}.$$

Suppose a, b are nonnegative extended real valued measurable functions on $\mathbb{R}^n$ such that $||a||_p = 1$ and $||b||_q = 1$.

Suppose $x \in \mathbb{R}^n$. By Hölder's Inequality we obtain

$$\begin{aligned} a * b(x) &= \int a(x-y)^{p/r} b(y)^{q/r} a(x-y)^{1-p/r} b(y)^{1-q/r}\, dy \\ &\leq \left(\int a(x-y)^p b(y)^q\, dy\right)^{1/r} \left(\int a(x-y)^p\, dy\right)^{1/q'} \left(\int b(y)^q\, dy\right)^{1/p'} \\ &= \left(\int a(x-y)^p b(y)^q\, dy\right)^{1/r}. \end{aligned}$$

By Tonelli's Theorem and the translation invariance of the Lebesgue integral we infer that

$$\begin{aligned}\int a * b(x)^r \int \left(\int a^p(x-y)b^q(y)\,dy\right)\,dx\\ &= \int \left(a^p(x-y)b^q(y)\,dx\right)\,dy \\ &= \int \left(a^p(x)b^q(y)\,dx\right)\,dy \\ &= \left(\int a^p(x)\,dx\right)\left(\int b(y)^q\,dy\right) \\ &= 1\end{aligned}$$

Thus the first inequality is proved.

Now suppose $f \in \mathbf{L}_p(\mathbb{R}^n)$ and $g \in \mathbf{L}_q(\mathbb{R}^n)$. Since $|f| * |g| \in \mathbf{L}_r(\mathbb{R}^n)$ by the first inequality we infer that $|f| * |g|(x) < \infty$ for almost all $x \in \mathbb{R}^n$ so $f * g$ is defined; moreover, for any $x \in \mathbb{R}^n$ we have

$$|f * g|(x) = |\int f(x-y)g(y)|\,dy \le \int |f(x-y)||g(y)|\,dy = |f| * |g|(x);$$

the second inequality now follows from the first. □

Theorem 4.0.4. Suppose $p \in [1,\infty]$, $f \in \mathbf{L}_p(\mathbb{R}^n)$ and $g \in \mathbf{L}_{p'}(\mathbb{R}^n)$. Then $E(f,g) = \emptyset$, $f * g$ is uniformly continuous and $\lim_{|x|\to\infty} f * g(x) = 0$.

Proof. That $E(f,g) = \emptyset$ follows directly from Hölder's Inequality and the translation invariance of the Lebesgue integral.

Suppose $x, a \in \mathbb{R}^n$. From Hölder's Inequality and the translation invariance of the Lebesgue integral we infer that

$$\begin{aligned}|f * g(x) - f * g(a)| &\le \int |f(x-y) - f(a-y)||g(y)|\,dy \\ &= \int |\tau_x(Af)(y) - \tau_a(Af)(y)||g(y)|\,dy \\ &\le ||\tau_x(Af) - \tau_a(Af)||_p||g||_{p'} \\ &\le ||\tau_x(Af) - \tau_a(Af)||_p||g||_{p'} \\ &\le ||\tau_{x-a}f - f||_p||g||_{p'};\end{aligned}$$

since

$$\lim_{h\to 0} ||\tau_h f - f||_p = 0$$

the uniform continuity of $f * g$ follows. (You remember this from real variable theory, right?)

To complete the proof we let

$$f_N = 1_{\mathbf{B}(0,N)\cap\{|f|\le N\}}f \quad \text{and we let} \quad g_N = 1_{\mathbf{B}(0,N)\cap\{|g|\le N\}}$$

for each positive integer N. Note that

$$f_N * g_N(x) = 0 \quad \text{if } |x| > 2N.$$

Moreover, by the Monotone Convergence Theorem,

$$\lim_{N \to \infty} ||f - f_N||_p + ||g - g_N||_{p'} = 0.$$

If $|x| > 2N$ we find that $f_N * g_N(x) = 0$ and we infer from Hölder's Inequality that

$$|f*g(x)| \le |(f-f_N)*g(x)| + |f*(g-g_N)(x)| \le ||f-f_N||_p ||g||_{p'} + ||f||_p ||g-g_N||;$$

the final assertion to be proved follows. □

4.1 Regularization.

Suppose $\phi \in \mathbf{L}_1(\mathbb{R}^n)$ and

$$\int \phi = 1.$$

For each $\epsilon > 0$ let $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$ for $x \in \mathbb{R}^n$ and note that

$$\int \phi_\epsilon = 1 \quad \text{and} \quad \int |\phi_\epsilon| = \int |\phi|.$$

Theorem 4.1.1. Suppose $1 \le p < \infty$ and $f \in \mathbf{L}_p(\mathbb{R}^n)$. Then

$$\lim_{\epsilon \downarrow 0} ||f * \phi_\epsilon - f||_p = 0.$$

Proof. For each $\epsilon > 0$ let

$$I_\epsilon = \int ||\tau_y f - f||_p |phi_\epsilon(y)| \, dy.$$

For each $x \in \mathbb{R}^n$ we have

$$|f*\phi_\epsilon(x) - f(x)| = \left| \int [f(x-y) - f(x)]\phi_\epsilon(y) \, dy \right| \le \int |f(x-y) - f(x)||\phi_\epsilon(y)| \, dy.$$

It follows from Minkowski's Inequality in Integral Form that

$$\begin{aligned} ||f * \phi_\epsilon - f||_p &\le \left(\int \left(\int |f(x-y) - f(x)||\phi_\epsilon(y)| \, dy \right)^p dx \right)^{1/p} \\ &\le \int \left(\int |f(x-y) - f(x)|^p |\phi_\epsilon(y)|^p \, dx \right)^{1/p} dy \\ &= I_\epsilon. \end{aligned}$$

For any $\delta > 0$ we have

$$\int_{\{|y|\le\delta\}} ||\tau_y f - f||_p |\phi_\epsilon(y)|\, dy \le \left(\sup_{|y|\le\delta} ||\tau_y f - f||_p\right) \int |\phi| \to 0 \quad \text{as} \quad \delta \to 0.$$

Moreover, for any $\delta > 0$,

$$\int_{\{|y|>\delta\}} ||\tau_y f - f||_p |\phi_\epsilon(y)|\, dy \le 2||f||_p \int_{|y|>\delta/\epsilon} \phi(y)\, dy \to 0 \quad \text{as} \quad \epsilon \downarrow 0.$$

□

Proposition 4.1.1. Suppose $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $f \in \mathbf{L}_1^{loc}(\mathbb{R}^n)$. Then $f * \phi_\epsilon \in \mathcal{E}(\mathbb{R}^n)$ for any $\epsilon > 0$ and

$$\lim_{\epsilon\downarrow 0} \int_K |f * \phi_\epsilon - f| = 0$$

for any compact subset K of $\mathbb{R}^n$. Moreover, if m is a nonnegative integer and f is m times continuously differentiable then

$$\partial^\alpha f * \phi_\epsilon = (\partial^\alpha f) * \phi_\epsilon$$

and

$$\lim_{\epsilon\downarrow 0} \sup\{|\partial^\alpha (f * \phi_\epsilon - f)(x)| : x \in K\} = 0$$

for any $\alpha \in \mathbf{M}(m, n)$.

Exercise 4.1.1. Prove the preceding Proposition.

There a many interesting ϕ's as above. One such is

$$\phi(x) = \mathcal{L}^n(\mathbf{B}(0,1))^{-1}(1 + |x|^2)^{(n+1)/2}, \quad x \in \mathbb{R}^n.$$

This is, in part, because

$$\mathbb{R}^n \times (\mathbb{R} \sim \{0\}) \ni (x, y) \mapsto \phi_y(x)$$

is harmonic in (x, y).

5 Generalized functions or distributions.

Whenever

$$u : \mathcal{D}(\Omega) \to \mathbb{C} \text{ is linear,}$$

m is a nonnegative integer and K is a compact subset of Ω we let

$$\sigma'_{m,K}(u) = \sup\{|u(\phi)| : \phi \in \mathcal{D}_K(\Omega) \text{ and } \sigma_{m,K}(\phi) \le 1\}$$

and note that

$$|u(\phi)| \leq \sigma'_{m,K}(u)\sigma_{m,K}(\phi) \quad \text{for } \phi \in \mathcal{D}_K(\Omega).$$

We let

$$\mathcal{D}'(\Omega)$$

be the vector space of linear functions $u : \mathcal{D}(\Omega) \to \mathbb{C}$ such that for each compact subset K of Ω there is a nonnegative integer m such that

$$\sigma'_{m,K}(u) < \infty.$$

Whenever $u \in \mathcal{D}'(\Omega)$ and G is an open subset of Ω we say u **vanishes on** G) if $u(\phi) = 0$ whenever $\phi \in \mathcal{D}(\Omega)$ and $\mathbf{spt}\, \phi \subset G$. We let

$$\mathbf{spt}\, u = \Omega \sim \cup\{G : G \text{ is an open subset of } \Omega \text{ and } u \text{ vanishes on } G\}$$

and call this closed subset of Ω the **support of** u. *This definition of support overrides the previously given definition of support.*

Exercise 5.0.2. Suppose $u \in \mathcal{D}(\prime)\Omega$. The $\mathbf{spt}\, u$ is the set of point $a \in \Omega$ such that for each open subset U of Ω such that $a \in U$ there is $\phi \in \mathcal{D}(\Omega)$ such that $\mathbf{spt}\, \phi \subset U$ and $u(\phi) \neq 0$.

Definition 5.0.1. Suppose $u \in \mathcal{D}'(\Omega)$. Let

$$\mathcal{E}_u(\Omega) = \{\phi \in \mathcal{E}(\Omega) : \mathbf{spt}\, \phi \cap \mathbf{spt}\, u \text{ is compact}\}.$$

Note that $\mathcal{D}(\Omega)$ and that $\mathcal{E}_u(\Omega)$ is a linear subspace of $\mathcal{E}(\Omega)$.

Lemma 5.0.1. Suppose $\phi \in \mathcal{E}_u(\Omega)$ and $\chi \in \mathcal{E}(\Omega)$ is such that $\mathbf{spt}\, u \cap \mathbf{spt}\, \phi \subset \mathbf{int}\, \{\chi = 1\}$. Then

$$\mathbf{spt}\, u \cap \mathbf{spt}\, (\phi - \chi\phi) = \emptyset.$$

Proof. We have

$$\mathbf{spt}\, (1 - \chi) = \mathbf{cl}\, \{\chi \neq 1\} = \mathbf{cl}\, \Omega \sim \{\chi = 1\} = \Omega \sim \mathbf{int}\, \{\chi = 1\}$$

so

$$\begin{aligned} \mathbf{spt}\, u \cap \mathbf{spt}\, (\phi - \chi\phi) &\subset \mathbf{spt}\, u \cap \mathbf{spt}\, \phi \cap \mathbf{spt}\, (1 - \chi) \\ &\subset (\mathbf{spt}\, u \cap \mathbf{spt}\, \phi) \sim \mathbf{int}\, \{\chi = 1\} \\ &= \emptyset. \end{aligned}$$

□

Proposition 5.0.2. There is one and only one linear map

$$v : \mathcal{E}_u \to \mathbb{C}$$

such that $v|\mathcal{D}(\Omega) = u$ and

$$v(\phi) = 0 \quad \text{whenever } \psi \in \mathcal{E}(\Omega) \text{ and } \mathbf{spt}\, \phi \cap \mathbf{spt}\, u = \emptyset.$$

Proof. Let

$$v = \{(\phi, u(\psi)) : \phi \in \mathcal{E}(\Omega),\ \psi \in \mathcal{D}(\Omega) \text{ and } \mathbf{spt}\, u \cap \mathbf{spt}\,(\phi - \psi) = \emptyset\}.$$

Suppose $(\phi, \psi_i) \in v$ for $i = 1, 2$. Then, as $\psi_2 - \psi_1 = (\psi_2 - \phi) - (\psi_1 - \phi)$

$$\mathbf{spt}\, u \cap \mathbf{spt}\,(\psi_2 - \psi_1) \subset \mathbf{spt}\, u \cap (\mathbf{spt}\,(\psi_2 - \phi) \cup (\mathbf{spt}\,(\psi_1 - \phi)) = \emptyset$$

so $0 = u(\psi_2 - \psi_1) = u(\psi_2) - u(\psi_1)$ so $u(\psi_2) = u(\psi_1)$. Thus v is a function.

It is a simple matter which we leave to the reader to verify that v is linear. It is obvious that v extends u.

Suppose $\phi \in \mathcal{E}_u(\Omega)$. Let $\chi \in \mathcal{D}(\Omega)$ be such that $\mathbf{spt}\, u \cap \mathbf{spt}\, \phi \subset \mathbf{int}\,\{\chi = 1\}$. By the preceding Lemma, $\mathbf{spt}\, u \cap \mathbf{spt}\,(\phi - \chi\phi) = \emptyset$. Since $\chi\phi \in \mathcal{D}(\Omega)$ we find that $\phi \in \mathbf{dmn}\, v$.

Finally, suppose $\tilde{v} : \mathcal{E}_u \to \mathbb{C}$ is a linear map such that $\tilde{v}|\mathcal{D}(\Omega) = u$ and

$$\tilde{v}(\phi) = 0 \quad \text{whenever } \psi \in \mathcal{E}(\Omega) \text{ and } \mathbf{spt}\, u \cap \mathbf{spt}\, \phi = \emptyset.$$

Suppose $\phi \in \mathcal{E}_u(\Omega)$. Choose $\chi \in \mathcal{D}(\Omega)$ such that $\mathbf{spt}\, u \cap \mathbf{spt}\, \phi \subset \mathbf{int}\,\{\chi = 1\}$. From the preceding Lemma we have $\mathbf{spt}\, u \cap \mathbf{spt}\,(\phi - \chi\phi) = \emptyset$. Thus, as $\chi\phi \in \mathcal{D}(\Omega)$,

$$\tilde{v}(\phi) = \tilde{v}(\chi\phi) + \tilde{v}(\phi - \chi\phi) = u(\chi\phi) + 0 = v(\phi)$$

so $\tilde{v} = v$. □

Remark 5.0.1. When convenient we shall identify u with the extension defined in the preceding Proposition.

The members of $\mathcal{D}'(\Omega)$ are called **generalized functions** or **distributions** on Ω. You will ask "In what way is the notion of generalized function a generalization of the notion of function?" We now proceed to embed $\mathbf{L}_1^{loc}(\Omega)$ in $\mathcal{D}'(\Omega)$ in a very useful way. We let

$$\iota : \mathbf{L}_1^{loc}(\Omega) \to \mathcal{D}'(\Omega)$$

be defined by

$$\iota(f)(\phi) = \int_\Omega f(x)\phi(x)\, dx \quad \text{whenever } \phi \in \mathcal{D}(\Omega).$$

Since

$$|\iota(f)(\phi)| \leq \left(\int_K |f|\right) \sigma_{0,K}(\phi)$$

whenever K is a compact subset of Ω and $\phi \in \mathcal{D}_K(\Omega)$ we find that $\iota(f)$ is indeed a generalized function. It is obvious that ι is linear and that

$$\mathbf{ker}\, \iota = \{f \in \mathbf{L}_1^{loc}(\Omega) : f(x) = 0 \text{ for almost all } x \in \Omega\}.$$

Definition 5.0.2. Suppose

$$T : \mathcal{D}(\Omega) \to \mathcal{D}(\tilde{\Omega})$$

is such that for every (k, K) such that $k \in \mathbb{N}$ and K is a compact subset of Ω there are (l, L) and $C \in [0, \infty)$ such that $l \in \mathbb{N}$, L is a compact subset of $\tilde{\Omega}$,

$$\mathbf{spt}\, \phi \subset L \quad \text{and} \quad \sigma_{l,L}(T\phi) \leq C\sigma_{k,K}(\phi) \quad \text{whenever } \phi \in \mathcal{D}_K(\Omega). \qquad (3)$$

It should be clear that we may define

$$T\prime : \mathcal{D}'(\tilde{)}\Omega \to \mathcal{D}'(\Omega)$$

by setting

$$T\prime u(\psi) = u(T\psi) \quad \text{whenever } \psi \in \mathcal{D}(\tilde{\Omega}).$$

In fact, if k, K, l, L, C are as above then

$$\sigma'_{m,K}(T') \leq C\sigma'_{l,L}(u).$$

We shall now define some extend some useful operations on functions to distributions in ways that will always be compatible with ι (whatever that means).

Suppose $\psi \in \mathcal{E}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. We let

$$\psi u(\phi) = u(\psi\phi) \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Exercise 5.0.3. Show that $\psi u \in \mathcal{D}'(\Omega)$. Show that

$$\phi\iota(f) = \iota(\phi f) \quad \text{whenever } f \in \mathbf{L}_1^{loc}(\Omega).$$

Remark 5.0.2. Multiplying generalized functions is rather problematical.

Suppose $F : \Omega \to \tilde{\Omega}$ is smooth diffeomorphism with range $\tilde{\Omega}$. Given $u \in \mathcal{D}'(\Omega)$ we let

$$Fu(\phi) = u((\phi \circ F)|\mathbf{det}\, \partial F|) \quad \text{for } \phi \in \mathcal{D}(\tilde{\Omega}).$$

Exercise 5.0.4. Show that $Fu \in \mathcal{D}'(\tilde{\Omega})$ and that, with appropriate hypotheses on G, $(G \circ F)u = G(Fu)$. Show that

$$F\iota(f) = \iota(Ff) \quad \text{whenever } f \in \mathbf{L}_1^{loc}(\Omega).$$

Definition 5.0.3. Suppose $\alpha \in \mathbf{M}(n)$ and $u \in \mathcal{D}'$. We let

$$\partial^\alpha u(\phi) = (-1)^{\mathbf{w}(\alpha)} u(\partial^\alpha \phi) \quad \text{whenever } \phi \in \mathcal{D}(\Omega).$$

Exercise 5.0.5. Show that $\partial^\alpha u \in \mathcal{D}'(\Omega)$. Show that if $f : \Omega \to \mathbb{C}$ is $\mathbf{w}(\alpha)$ times continuously differentiable on Ω then

$$\partial^\alpha \iota(f) = \iota(\partial^\alpha f).$$

Definition 5.0.4. We say $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is a **linear partial differential operator (of order** m**)** for each $\alpha \in \mathbf{M}(n)$ with $\mathbf{w}(\alpha) \le m$ there are $p_\alpha \in \mathcal{E}(\Omega)$ such that

$$Pu = \sum_{\mathbf{w}(\alpha)\le m} p_\alpha \partial^\alpha u \quad \text{whenever } u \in \mathcal{D}'(\Omega).$$

Exercise 5.0.6. Suppose $u \in \mathcal{D}'(\Omega)$ and $j \in \{1, \ldots, n\}$. Show that

$$\partial_j u(\phi) = \lim_{t\to 0} \frac{1}{t}(\tau_{-t\mathbf{e}_j} u - u)(\phi) \quad \text{whenever } \phi \in \mathcal{D}(\Omega).$$

Hint: Use Taylor's Theorem to write

$$\frac{1}{t}(\tau_{-t\mathbf{e}_j}\phi - \phi) = t\psi_t$$

where $\psi_t(x) = \phi^{<1>}(x, t\mathbf{e}_j)(\mathbf{e}_j)$ for appropriate t, x.

Definition 5.0.5. Suppose $u \in \mathcal{D}'(\Omega)$, Γ is an open subset of some Euclidean space,

$$F : \Omega \to \Gamma \quad \text{is smooth}$$

and

$$F^{-1}[K] \cap \mathbf{spt}\, u \quad \text{is compact for any compact subset } K \text{ of } \Gamma.$$

We let

$$F_\# u(\psi) = u(\psi \circ F) \quad \text{for } \psi \in \mathcal{D}(\Gamma).$$

Exercise 5.0.7. Show that $F_\# u \in \mathcal{D}'(\Gamma)$. Supply the hypotheses that make the assertion

$$(G \circ F)_\# u = G_\#(F_\# u)$$

true.

Note that difference between Fu and $F_\# u$ when F is a diffeomorphism.

5.1 The Dirac delta function.

Definition 5.1.1. Suppose $a \in \Omega$. The **delta function at** a is that member δ_a of $\mathcal{E}'(\Omega)$ such that

$$\delta_a(\phi) = \phi(a) \quad \text{whenever } \phi \in \mathcal{E}(\Omega).$$

Evidently, δ_a is a complex valued function on $\mathcal{E}(\Omega)$ and, since

$$|\delta_a(\phi)| \le |\phi(a)| = \sigma_{0,\{a\}}(\phi) \quad \text{for } \phi \in \mathcal{E}(\Omega),$$

we find that $\delta_a \in \mathcal{E}'(\Omega)$.

Remark 5.1.1. Of course the delta function is not a function; it's a generalized function.

Theorem 5.1.1. Suppose $a \in \Omega$, $u \in \mathcal{D}'(\Omega)$,

$$\mathbf{spt}\, u \subset \{a\},$$

m is a nonnegative integer, K is a compact subset of Ω such that $a \in \mathbf{int}\, K$ and

$$\sigma'_{m,K}(u) < \infty.$$

Then

$$u = \sum_{\mathbf{w}(\alpha) \le m} (-1)^{\mathbf{w}(\alpha)} u(\tau_{-a} m_\alpha) \partial^\alpha \delta_a.$$

Proof. We may assume without loss of generality that $a = 0$. Suppose $\phi \in \mathcal{D}_K(\Omega)$. From Taylor's Theorem we have

$$\phi = \sum_{\mathbf{w}(\alpha) \le m} \partial^\alpha u(0) m_\alpha + r$$

where $r(x) = \phi^{<m+1>}(0, x)(x)$ whenever $x \in \Omega$ and $(0, x) \in < \Omega >$.

From Leibniz' Rule we infer that there is $C \in [0, \infty)$ such that

$$|\partial^l r(x)| \le C|x|^{(m+1)-l} \quad \text{whenever } x \in K \text{ and } 0 \le l \le m. \tag{4}$$

Let $\psi \in \mathcal{D}(\Omega)$ be such that $0 \in \mathbf{int}\, \{\psi = 1\}$ and, for each $\epsilon \in (0, 1)$, let $\psi_\epsilon(x) = \psi(\epsilon^{-1} x)$ for $x \in \epsilon\Omega$.

Suppose $\alpha \in \mathbf{M}(n)$ and $\mathbf{w}(\alpha) \le m$. Then

$$(-1)^{\mathbf{w}(\alpha)} u(m_\alpha)(\partial^\alpha \delta_0)(\phi) = u(m_\alpha)\delta_0)(\partial^\alpha \phi) = u(\partial^\alpha \phi(0) m_\alpha).$$

Thus

$$u(\phi) - \sum_{\mathbf{w}(\alpha) \le m} (-1)^{\mathbf{w}(\alpha)} u(m_\alpha) \partial^\alpha \delta_0 = u(r) = u(\psi_\epsilon r).$$

But whenever $0 \le l \le m$ we have

$$|\partial^l (\psi_\epsilon r)| = \left| \sum_{0 \le k \le l} \binom{l}{k} (\partial^k \psi_\epsilon)(\partial^{l-k} r) \right| \le C \sigma_{m,K}(\psi) \sum_{k \le l} \binom{l}{k} \epsilon^{-k)} \epsilon^{(m+1)-(l-k)}$$

so that

$$\lim_{\epsilon \downarrow 0} \sigma_{m,K}(\psi_\epsilon r) = 0.$$

□

5.2 Homogeneity.

This concept is *extremely* useful. Suppose $p \in \mathbb{R}$. We let

$$E \in \mathcal{X}(\mathbb{R}^n)$$

be such that $E(x) = x$ for $x \in \mathbb{R}^n$; we call E the **Euler vector field**. Note that
$$\frac{d}{dt} e^t x = E(e^t x) \quad \text{whenever } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n.$$

Suppose $f : \mathbb{R}^n \sim \{0\} \to \mathbb{C}$. We say f is **homogeneous of degree** p if
$$f(tx) = t^p f(x) \quad \text{whenever } 0 < t < \infty \text{ and } x \in \mathbb{R}^n \sim \{0\}.$$

Equivalently,
$$\mu_{1/t} f = t^p f \quad \text{whenever } 0 < t < \infty.$$

Proposition 5.2.1. Suppose $f \in \mathcal{E}(\mathbb{R}^n \sim \{0\})$. Then f is homogeneous of degree p if and only if $Ef = pf$.

Exercise 5.2.1. Prove the preceding Proposition.

Definition 5.2.1. Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$. We say f is homogeneous of degree p if
$$\mu_{1/t} u = t^p u \quad \text{whenever } 0 < t < \infty.$$

Proposition 5.2.2. Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$. Then u is homogeneous of degree p if and only if $Eu = pu$.

Exercise 5.2.2. Prove the preceding Proposition.

Exercise 5.2.3. Show that the delta function δ_0 is homogeneous of degree $-n$. Show that if $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree p and $\alpha \in \mathbf{M}(m, n)$ then $\partial^\alpha u$ is homogeneous of degree $p - m$.

5.3 Integration against a smooth family.

Suppose m is a positive integer, Γ is an open subset of $\mathbb{R}^m$, $\chi \in \mathcal{E}(\Omega \times \Gamma)$,
$$\{y \in \Gamma : (x, y) \in \mathbf{spt}\, u\} \quad \text{is compact for any } x \in \Omega$$

and $u \in \mathcal{D}'(\Gamma)$. We define
$$u_\chi : \Omega \to \mathbb{C}$$

by letting
$$u_\chi(x) = u(\chi \circ i_x) \quad \text{whenever } x \in \Omega;$$

here, for each $x \in \Omega$, we let $i_x : \Gamma \to \Omega \times \Gamma$ carry $y \in \Gamma$ to (x, y). Alternatively,
$$u_\chi(x) = u(\Gamma \ni y \mapsto \chi(x, y)) \quad \text{for } x \in \Omega.$$

The following Proposition is an immediate consequence of this definition.

Proposition 5.3.1. Suppose $f \in \mathbf{L}_1^{loc}(\Gamma)$. Then
$$\iota(f)_\chi = \iota\left(\Omega \ni x \mapsto \int_\Gamma \chi(x, y) f(y)\, dy\right).$$

Theorem 5.3.1. $u_\chi \in \mathcal{E}(\Omega)$. Moreover,

$$\partial_j(u_\chi) = u_{\partial_{(j,0)}\chi} \quad \text{for any } j \in \{1, \ldots, m\}.$$

Proof. From Taylor's Theorem we obtain for each $j = 1, \ldots, o$ and each $(x, h) \in < \Omega >$ a function $r_{j,x,h} \in \mathcal{E}(\Gamma)$ such that

$$\chi(x+h, y) - \chi(x, y) - \sum_{j=1}^{m} h_j \partial_{(j,0)}\chi(x, y) = \sum_{j=1}^{m} h_j r_{j,x,h}(y)$$

whenever $(x, h) \in < \Omega >$ and $y \in \Gamma$;

$$r_{j,x,0}(x) = 0$$

and

$$< \Omega > \times \Gamma \ni ((x, h), y) \mapsto r_{j,x,h}(y) \in \mathcal{E}(< \Omega > \times \Gamma);$$

It follows that for any nonnegative integer l, any compact subset K of Ω and any compact subset L of Γ we have

$$\lim_{h \to 0} \sup\{|\sigma_{l,L}(r_{j,x,h})| : x \in K\} = 0.$$

Since

$$u_\chi(x+h) - u_\chi(x) - \sum_{j=1}^{m} h_j u_{\partial_{(j,0)}\chi}(x) = \sum_{j=1}^{m} h_j u(r_{j,x,h})$$

the Theorem follows. □

Theorem 5.3.2. Suppose $\psi \in \mathcal{D}(\Omega)$. Then

$$\int u_\chi(x)\psi(x)\,dx = u\left(\Gamma \ni y \mapsto \int \chi(x, y)\psi(x)\,dx\right).$$

Proof. Let $\zeta(y) = \int_\Omega \chi(x, y)\psi(x)\,dx$ for $y \in \Gamma$. For each $\delta > 0$ let

$$\zeta_\delta(y) = \delta^{-n} \sum_{z \in \mathbb{Z}^n,\ \delta z \in \Omega} \chi(\delta z, y)\psi(\delta z) \quad \text{whenever } y \in \Gamma.$$

Evidently, for any compact subset L of Γ and any nonnegative integer l we have $\sigma_{l,L}(\zeta - \zeta_\delta) \to 0$ as $\delta \downarrow 0$. Keeping in mind that u_χ is continuous we have

$$u(\zeta_\delta) = \delta^{-n} \sum_{z \in \mathbb{Z}^n} u_\chi(\delta z)\psi(\delta z) \to \int_\Omega u_\chi(x)\psi(x)\,dx \quad \text{as } \delta \downarrow 0;$$

the Theorem follows. □

5.4 Smoothing distributions.

Suppose

$$\phi \in \mathcal{D}(\mathbb{R}^n), \quad \int \phi = 1 \quad \text{and} \quad \mathbf{spt}\, \phi \subset \mathbf{U}(n,0)1.$$

For each $\epsilon > 0$ let $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$ for $x \in \mathbb{R}^n$ and note that $\int \phi_\epsilon = 1$. Let

$$\Phi_\epsilon(x,y) = \phi_\epsilon(x-y) \quad \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Remark 5.4.1. Suppose $f \in \mathbf{L}_1^{loc}(\mathbb{R}^n)$ and $\epsilon > 0$. Then

$$\iota(f)_\epsilon(x) = \iota(f)(\mathbb{R}^n \ni y \mapsto \phi_\epsilon(x-y)) = \int f(y)\phi_\epsilon(x-y)\,dy = (\phi_\epsilon * f)(x)$$

for any $x \in \mathbb{R}^n$.

Theorem 5.4.1. Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ and, for each $\epsilon > 0$,

$$u_\epsilon = u_{\Phi_\epsilon}.$$

Then

$$\mathbf{spt}\, u_\epsilon \subset \{x \in \mathbb{R}^n : \mathbf{dist}\,(x, \mathbf{spt}\, u) \le \epsilon\},$$
$$\lim_{\epsilon \downarrow 0} \iota(u_\epsilon)(\psi) = u(\psi) \quad \text{whenever } \psi \in \mathcal{D}(\mathbb{R}^n)$$

and

$$\partial_j u_\epsilon = (\partial_j u)_\epsilon \quad \text{whenever } j \in \{1,\ldots,n\}.$$

Proof. Suppose $\psi \in \mathcal{D}(\mathbb{R}^n)$. Using Theorem **??** we find that

$$\begin{aligned}
\iota(u_\epsilon)(\psi) &= \int u_\epsilon(x)\psi(x)\,dx \\
&= \int u(\mathbb{R}^n \ni y \mapsto \phi_\epsilon(x-y))\psi(x)\,dx \\
&= u\left(\mathbb{R}^n \ni y \mapsto \int \phi_\epsilon(x-y)\psi(x)\,dx\right) \\
&= u((A\phi_\epsilon) * \psi) \\
&\to u(\psi),
\end{aligned}$$

as desired.

Finally, let $j \in \{1,\ldots,n\}$ and note that

$$\partial_{(j,0)}\Phi_\epsilon = -\partial_{(0,j)}\Phi_\epsilon.$$

Then for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
\partial_j(u_\epsilon)(x) &= \partial_j(u_{\Phi_\epsilon})(x) \\
&= u_{\partial_{(j,0)}\Phi_\epsilon}(x) \\
&= -u_{\partial_{(0,j)}\Phi_\epsilon}(x) \\
&= -u(\mathbb{R}^n \ni y \mapsto \partial_{(0,j)}\Phi_\epsilon(x,y)) \\
&= (\partial_j u)(\mathbb{R}^n \ni y \mapsto \Phi_\epsilon(x,y)) \\
&= (\partial_j u)_\epsilon(x).
\end{aligned}$$

□

Theorem 5.4.2 (The constancy Theorem)**.** Suppose Ω is connected, $u \in \mathcal{D}'(\Omega)$ and $\partial_j u = 0$ for $j = 1, \ldots, n$. Then there is $c \in \mathbb{R}$ such that

$$u(\psi) = c \int_\Omega \psi \quad \text{whenever } \psi \in \mathcal{D}(\Omega).$$

Remark 5.4.2. That is, u is the generalized function which equals ι applied to the function which equals c at each point of Ω

Proof. Suppose $a \in \Omega$. Choose $R \in (0, \infty)$ such that $\mathbf{B}(a, R) \subset \Omega$. Let $v \in \mathcal{D}(\mathbb{R}^n)$ be such that $v(\psi) = u(\psi|\Omega)$ whenever $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\mathbf{spt}\, \psi \subset \mathbf{U}(a, R)$. Then $\partial_j v$ vanishes on $\mathbf{U}(a, R)$ for each $j = 1, \ldots, n$. Let v_ϵ, $\epsilon > 0$, be a smoothing of v where ϕ as above is supported in $\mathbf{B}(0, 1)$. Since $\partial_j(v_\epsilon) = (\partial_j v)_\epsilon$ we find that, whenever $0 < \epsilon < R$, $\partial_j(v_\epsilon)$ is zero on $\mathbf{U}(a, R-\epsilon)$ so there is $c_\epsilon \in \mathbb{C}$ such that $v_\epsilon = c_\epsilon$ on $\mathbf{U}(a, R - \epsilon)$. Since $v_\epsilon(\psi) \to v(\psi)$ whenever $\psi \in \mathcal{D}(\mathbb{R}^n)$ we infer that there is c such that $u(\psi) = v(\psi) = c \int \psi$ whenever $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\mathbf{spt}\, \psi \subset \mathbf{U}(a, R)$. Since Ω is connected the Theorem follows. □

5.5 Integration over the fiber.

Suppose m is a positive integer, Γ is an open subset of $\mathbb{R}^m$ and

$$F : \Omega \to \Gamma$$

is a submersion; this means by definition, that F is smooth and

$$\mathbf{rng}\, \partial F(x) = \mathbb{R}^m \quad \text{whenever } x \in \Omega.$$

Note that this implies $n \geq m$. Set

$$J_m F(x) = |\textstyle\bigwedge_m \partial F(x)| = \sqrt{\mathbf{det}\, \partial F(x) \circ \partial F(x)^*} \quad \text{for } x \in \Omega.$$

For any $\chi \in \mathcal{D}(\Omega)$ one has the *coarea formula*

$$\int_\Omega \chi(x) J_m F(x)\, dx = \int_\Gamma \left(\int_{F^{-1}[\{y\}]} \phi(x)\, d\mathcal{H}^{n-m} x \right) dy.$$

(Can you prove it?)

For each $\chi \in \mathcal{D}(\Omega)$ let

$$\chi_F(y) = \int_{F^{-1}[\{y\}]} \chi(x) J_m F(x)^{-1}\, d\mathcal{H}^{n-m} x \quad \text{for } y \in \Gamma$$

and note that $\chi_F \in \mathcal{D}(\Gamma)$ since $\mathbf{spt}\, \chi_F \subset F[\mathbf{spt}\, \chi]$.

For each $v \in \mathcal{D}'(\Gamma)$ we define

$$v \circ F \in \mathcal{D}'(\Omega)$$

by setting

$$(v \circ F)(\chi) = v(\chi_F) \quad \text{for } \chi \in \mathcal{D}(\Omega).$$

Suppose $f \in \mathbf{L}_1^{loc}(\Omega)$. Then, by the coarea formula,

$$\begin{aligned} \iota(f) \circ F(\chi) &= \int_\Gamma f(y) \left(\int_{F^{-1}[\{y\}]} \chi(x) J_m F(x)^{-1} \, d\mathcal{H}^{n-m} x \right) dy \\ &= \int_\Gamma \left(\int_{F^{-1}[\{y\}]} f(F(x))\chi(x) J_m F(x)^{-1} \, d\mathcal{H}^{n-m} x \right) dy \\ &= \int f(F(x))\chi(x) \, dx \\ &= \iota(f \circ F)(\chi); \end{aligned}$$

This motivates the notation.

Let

$$\mathcal{X}_F(\Omega)$$

be the vector space of those $X \in \mathcal{X}(\Omega)$ such that

$$\partial F(x)(X(x)) = 0 \quad \text{for } x \in \Omega$$

which is to say that X is tangent to the fibers of F.

Theorem 5.5.1. Suppose $v \in \mathcal{D}'(\Gamma)$. Then

$$X(v \circ F) = 0 \quad \text{whenever } X \in \mathcal{X}_F(\Omega).$$

Suppose $u \in \mathcal{D}'(\Omega)$ and

$$X(u) = 0 \quad \text{whenever } X \in \mathcal{X}_F(\Omega).$$

Then $u = v \circ F$ for some $v \in \mathcal{D}'(\Gamma)$.

Exercise 5.5.1. Prove the preceding Theorem. It suffices to consider the case when $\Omega = A \times B$, $\Gamma = B$ and $F(x, y) = y$ for $(x, y) \in A \times B$.

Exercise 5.5.2. Suppose $u \in \mathcal{D}'(\mathbb{R}^2)$ and $S(x, t) = x - t$ for $(x, t) \in \mathbb{R}^2$. Show that

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

if and only if $u = v \circ S$ for some $v \in \mathcal{D}'(\mathbb{R})$.

6 Products

Definition 6.0.1 (Tensor product functions)**.** Suppose N is a positive integer and for each $j = 1, \ldots, N$, X_j is a set and $\phi_j : X_j \to \mathbb{C}$. We define the **tensor product**

$$\phi_1 \otimes \cdots \otimes \phi_N : X_1 \times \cdots \times X_N \to \mathbb{C}$$

be such that

$$\phi_1 \otimes \cdots \otimes \phi_N(x_1, \ldots, x_N) = \prod_{j=1}^{N} \phi_j(x_j)$$

whenever $(x_1, \ldots, x_N) \in X_1 \times \cdots \times X_N$.

Definition 6.0.2. We let

$$\mathcal{P}(\Omega)$$

be the linear subspace of $\mathcal{D}(\Omega)$ spanned by the functions $\psi \in \mathcal{D}(\Omega)$ corresponding to which there exist $\phi_j \in \mathcal{D}(\mathbb{R})$ such that

$$\psi = (\phi_1 \otimes \cdots \otimes \phi_n)|\Omega.$$

Theorem 6.0.2. Suppose $\chi \in \mathcal{D}(\Omega)$. For each nonnegative integer m, each compact subset K of Ω such that $\mathbf{spt}\,\chi \subset \mathbf{int}\,K$ and each $\eta > 0$ there is $\zeta \in \mathcal{D}_K(\Omega) \cap \mathcal{P}(\Omega)$ such that

$$\sigma_{m,K}(\chi - \zeta) < \eta.$$

Proof. Let R be a positive real number such that

$$\prod_{j=1}^{n}(a_j - R, a_j + R) \subset \Omega \quad \text{whenever } x \in \Omega.$$

Let $\psi \in \mathcal{D}(R)$ be such that $\mathbf{spt}\,\psi \subset (-1, 1)$ and $\int_{\mathbb{R}} \psi(t)\,dt = 1$. For each $\epsilon \in (0, R)$ let

$$\phi_\epsilon(x) = \epsilon^{-n} \prod_{j=1}^{n} \psi(x_j/\epsilon) \quad \text{for } x \in \Omega;$$

note that $\phi_\epsilon \in \mathcal{D}(\Omega) \cap \mathcal{P}(\Omega)$ and that $\int_\Omega \phi_\epsilon(x)\,dx = 1$.

For each $\epsilon \in (0, R)$ let

$$\chi_\epsilon(x) = \int \phi_\epsilon(x - y)\chi(y)\,dy \quad \text{for } x \in \Omega$$

and note that $\chi_\epsilon \in \mathcal{D}(\Omega)$. Making the substitution $y \mapsto x - y$ in the integral defining χ_ϵ we find that

$$\chi_\epsilon(x) = \int \chi(x - y)\phi_\epsilon(y)\,dy$$

which implies that

$$\partial^\alpha \chi_\epsilon(x) = \int \partial^\alpha \chi(x - y)\phi_\epsilon(y)\,dy$$

for any $\alpha \in \mathbf{M}(n)$; since $\int \phi_\epsilon = 1$ we infer that

$$\partial^\alpha(\chi - \chi_\epsilon)(x) = \int (\partial^\alpha \chi(x - y) - \partial^\alpha \chi(x))\phi_\epsilon(y)\,dy.$$

Thus

$$\lim_{\epsilon \downarrow 0} \sigma_{m,K}(\chi - \chi_\epsilon) = 0 \quad \text{for any } m \in \mathbb{N}.$$

Whenever $\delta, \epsilon \in (0, R)$ we let

$$\chi_{\delta,\epsilon}(x) = \delta^{-n} \sum_{z \in \mathbb{Z}^n} \chi(\delta z)\psi_\epsilon(x - \delta z) \quad \text{whenever } x \in \Omega;$$

note that $\chi_{\delta,\epsilon} \in \mathcal{P}(\Omega) \cap \mathcal{D}_K(\Omega)$.

Let $C = \{x \in \mathbb{R}^n : 0 \le x_j < 1,\ j = 1, \ldots, n\}$. Suppose $\alpha \in \mathbf{M}(n)$ and $x \in \mathbb{R}^n$. Then

$$\partial^\alpha(\phi_\epsilon * \chi - \chi_{\delta,\epsilon})(x) = \sum_{z \in \mathbb{Z}^n} \int_{\delta(z+C)} \partial^\alpha \phi_\epsilon(x - z)\chi(z) - \partial^\alpha \phi_\epsilon(x - \delta z)\chi(\delta z)\, dz.$$

Finally,

$$(\chi - \chi_\epsilon)(x) = (\chi - \chi * \phi_\epsilon)(x) = \int_{\mathbb{R}^n} (\chi(x) - \chi(x - y))\phi_\epsilon(x - y)\, dy$$

so

$$\partial^\alpha(\chi - \phi_\epsilon * \chi)(x) = \int_{\mathbb{R}^n} (\partial^\alpha \chi(x) - \partial^\alpha \chi(x - y))\phi_\epsilon(x - y)\, dy.$$

which tends to zero uniformly as $\delta \downarrow 0$. □

Theorem 6.0.3. Suppose A is an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}'(A)$, B is an open subset of $\mathbb{R}^o$ and $v \in \mathcal{D}'(B)$. Then there is one and only one

$$u \times v \in \mathcal{D}'(A \times B)$$

such that

$$(u \times v)(\phi \otimes \psi) = u(\phi)v(\psi) \tag{5}$$

whenever $\phi \in \mathcal{D}(A)$, $\psi \in \mathcal{D}(B)$ and where we have set

$$(\phi \otimes \psi)(x, y) = \phi(x)\psi(y) \quad \text{whenever } (x, y) \in A \times B.$$

Moreover, if $\chi \in \mathcal{D}(A \times B)$ then

$$(u \times v)(\chi) = u(v_\chi). \tag{6}$$

Proof. The uniqueness follows directly from the preceding Theorem.

To prove existence we let $u \times v : \mathcal{D}(A \times B) \to \mathbb{C}$ be defined by (**??**). It is evident that $u \times v$ is linear. Moreover, if $\phi \in \mathcal{D}(A)$ and $\psi \in \mathcal{D}(B)$ then

$$u(v_{\phi \otimes \psi}) = u(A \ni x \mapsto v(B \ni y \mapsto \phi(x)\psi(y))) = u(\phi)v(\psi)$$

so (**??**) holds.

Suppose $\chi \in \mathcal{D}(A \times B)$, K is a compact subset of $A \times B$ and $\mathbf{spt}\, \chi \subset K$. Let $L = \{x : (x, y) \in K\}$, note that L is a compact subset of A, let $M = \{y :$

$(x,y) \in K\}$ and note that M is a compact subset of Y. If m is a nonnegative integer we have

$$|(u \times v)(\chi)| = |u(v_\chi)| \le \sigma'_{m,L}(u)\sigma_{m,L}(v_chi)$$

and, for any $\alpha \in \mathbf{M}(n)$ with $\mathbf{w}(\alpha) \le m$ and any $x \in L$,

$$|\partial^\alpha(v_\chi)(x)| = |(\partial^\alpha v)_\chi(x)| \le \sigma'_{l,M}(\partial^\alpha)\sigma_{l,M}(\chi).$$

It follows that $u \times v \in \mathcal{D}'(A \times B)$. □

Definition 6.0.3. Suppose $u \in \mathcal{D}'(A)$, $v \in \mathcal{D}'(B)$ and

$$\{(x,y) \in \mathbf{spt}\, u \times \mathbf{spt}\, v : x + y = z\} \text{ is compact whenever } z \in \mathbb{R}^n.$$

Let

$$u * v = A_\#(u \times v).$$

One may easily verify the properties of convolution, such as commutativity and associativity. Suppose $T(x,y) = (y,x)$ for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then

$$\begin{aligned} T_\#(u \times v)(\phi \otimes \psi) &= (u \times v)((\phi \otimes \psi) \circ T) \\ &= (u \times v)(\psi \otimes \phi) \\ &= u(\psi)v(\phi) \\ &= (v \times u)(\phi \otimes \psi); \end{aligned}$$

our characterization of products now implies that

$$T_\#(u \times v) = v \times u.$$

Since $A \circ T = A$ we find that

$$\begin{aligned} u * v &= A_\#(u \times v) \\ &= (A \circ T)_\#(u \times v) \\ &= A_\#(T_\#(u \times v)) \\ &= A_\#(v \times u) \\ &= v * u. \end{aligned}$$

Now let $I(z) = z$ for $z \in \mathbb{R}^n$ and let $C(x,(y,z)) = ((x,y),z)$ for $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. We use (**??**) to infer that

$$C_\#(u \times (v \times w)) = (u \times v) \times w.$$

Since

$$A \circ (A \times I) \circ C = A \circ (I \times A)$$

we find that

$$\begin{aligned}(u * v) * w &= A_{\#}((u * v) \times w) \\ &= A_{\#}(A_{\#}(u \times v) \times I_{\#} w) \\ &= A_{\#}((A \times I)_{\#}((u \times v) \times w) \\ &= A_{\#}(A \times I)_{\#} C_{\#}(u \times (v \times w)) \\ &= A_{\#}(I \times A)_{\#}(u \times v) \times w) \\ &= A_{\#}(u \times (v * w)) \\ &= u * (v * w).\end{aligned}$$

Proposition 6.0.1. Suppose $u \in \mathcal{D}'(A)$, $v \in \mathcal{D}'(B)$ and $a \in \mathbb{R}^n$. Then

$$(\tau_a u) * v = \tau_a(u * v) = u * (\tau_a v).$$

Moreover,

$$(\partial^\alpha u) * v = \partial^\alpha (u * v) = u * (\partial^\alpha v) \quad \text{for any } \alpha \in \mathbf{M}(n).$$

Exercise 6.0.3. Prove the preceding Proposition.