The critical set goes to a set of measure zero.

**Proposition.** Suppose \( S \) is an \( r \)-dimensional linear subspace of \( \mathbb{R}^n \) and \( b \in \mathbb{R}^n \). Then for any \( R > 0 \) and any \( \epsilon > 0 \) we have
\[
\mathcal{L}^n(\{y \in B_b(R) : \text{dist} (y, b + S) \leq \epsilon\}) \leq 2^r R^r \epsilon^{n-r}.
\]

**Proof.** Translating by \(-b\) if necessary we may assume that \( b = 0 \). Rotating \( S \) if necessary we may assume that \( S = \mathbb{R}^r \times \{0\} \) where we identify \( \mathbb{R}^n \) with \( \mathbb{R}^r \times \mathbb{R}^{n-r} \) and where we make use of the fact that
\[
\mathcal{L}^n(L[A]) = |\det L|\mathcal{L}^n(A) \quad \text{for any Lebesgue measurable subset of } \mathbb{R}^n.
\]

Finally, we observe that
\[
\mathcal{L}^n(\{(y, z) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : |(y, z)| \leq R \text{ and } |z| \leq \epsilon\})
\leq \mathcal{L}^n(\{(y, z) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : |y| \leq R \text{ and } |z| \leq \epsilon\})
\leq 2^n R^r \epsilon^{n-r}
\]
by Tonelli’s Theorem. \( \Box \)

**Corollary.** Suppose
\begin{enumerate}
\item \( U \) is an open subset of \( \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \) is continuously differentiable;
\item \( a \in A \subset U \) and \( A \) is convex;
\item \( M = \sup\{||\partial f(x)|| : x \in A\} < \infty; \)
\item \( \epsilon = \sup\{||\partial f(x) - \partial f(a)|| : x \in A\} < \infty; \) and
\item \( r = \dim \operatorname{range} \partial f(a). \) Then
\item \( \mathcal{L}^n(\partial f[A]) = 2^n (\dim \mathcal{A})^n M^r \epsilon^{n-r}. \)
\end{enumerate}

**Proof.** Let \( S \) be the range of \( \partial f(a) \) and let \( b = f(a) \). We will show that
\[
f[A] \subset \{y \in B_b(\max \mathcal{A}) : \text{dist} (y, b + S) \leq \epsilon \max \mathcal{A}\}
\]
and invoke the previous Proposition. By the Fundamental Theorem of Calculus we have
\[
f(x) = b + \int_0^1 \partial f(a)(a + t(x - a))(x - a) \, dt
\]
for any \( x \in A \) from which it follows that
\[
f[A] \subset B_b(\max \mathcal{A})
\]
We may rewrite (6) as
\[
f(x) = b + \partial f(a)(x - a) + \int_0^1 [\partial f(a)(a + t(x - a)) - \partial f(a)(a)](x - a) \, dt
\]
for any \( x \in A \); it follows that
\[
|(f(x) - b) \cdot w| \leq \epsilon \max \mathcal{A}|w|
\]
for any \( x \in A \) and any \( w \in S^\perp \) which implies that the distance from \( f(x) \) to \( S \) does not exceed \( \epsilon \max \mathcal{A} \). \( \Box \)

**Standard Cubical Subdivision.** Let
\[
C = \{x \in \mathbb{R}^n : 0 \leq x_j < 1 \text{ for } j = 1, \ldots, n\}.
\]
For each integer \( m \) let
\[
C_m = \{2^{-m}(z + C) : z \in \mathbb{Z}^n\}.
\]
Note that \( \text{diam} C = \sqrt{n2^{-m}} \) whenever \( C \in \mathcal{C}_m \).

**Theorem.** Suppose \( U \) is an open subset of \( \mathbb{R}^n \),
\[ f : U \to \mathbb{R}^n \]
continuously differentiable and
\[ A = \{ x \in U : \text{dim} \text{rng} \partial f(x) < n \}. \]
Then
\[ \mathcal{L}^n(f[A]) = 0. \]

**Proof.** For each \( k = 1, 2, 3, \ldots \) let
\[ A_k = \{ x \in U \cap B_k(0) : \text{dim} \text{rng} \partial f(x) < n \text{ and dist (} x, \mathbb{R}^n \sim A \} \geq \frac{1}{k} \} \]
and note that \( A_k \) is compact. Since \( A = \bigcup_{k=1}^{\infty} A_k \) it will suffice to show that \( \mathcal{L}^n(f[A_k]) = 0 \) for each \( k \).

Let \( k \) be a positive integer. Choose an open set \( G \) such that \( A_k \subset G \) and \( \text{cl} G \) is a compact subset of \( U \). Then \( M = \max \{ ||\partial f(x)|| : x \in \text{cl} G \} < \infty \).

Suppose \( 0 < \epsilon \leq M \). Let \( \mathcal{U} \) be the family of those open balls \( U_a(s) \) which are subsets of \( G \) and are such that \( a \in A_k \) and
\[ ||\partial f(x) - \partial f(a)|| \leq \epsilon \quad \text{whenever} \quad x \in U_a(s); \]
Evidently, \( \mathcal{U} \) is an open covering of \( A_k \); let \( \rho \) be its Lebesgue number. Next choose a positive integer \( N \) such that \( \sqrt{n2^{-M}} < \rho \). Let \( \mathcal{D} \) be the family of those \( C \in \mathcal{C}_N \) such that \( C \cap A_k \neq \emptyset \). By the preceding Corollary,
\[ \mathcal{L}^n(f[C]) \leq 2^n(\text{diam} C)^n M^{n-1} \epsilon = (2\sqrt{n})^n M^{n-1} \epsilon \mathcal{L}^n(C) \]
for any \( C \in \mathcal{D} \). Thus
\[ \mathcal{L}^n(f[A_k]) \leq \sum_{C \in \mathcal{D}} \mathcal{L}^n(f[C]) \leq \sum_{C \in \mathcal{D}} (2\sqrt{n})^n M^{n-1} \epsilon \mathcal{L}^n(C) = (2\sqrt{n})^n M^{n-1} \epsilon \mathcal{L}^n(\bigcup \mathcal{D}) \leq (2\sqrt{n})^n M^{n-1} \epsilon \mathcal{L}^n(G). \]
Owing to the arbitrariness of \( \epsilon \) we may conclude that \( \mathcal{L}^n(f[A_k]) = 0. \)