The critical set goes to a set of measure zero.

Proposition. Suppose S is an r-dimensional linear subspace of $\mathbf{R}^n$ and $\mathbf{b} \in \mathbf{R}^n$. Then for any $R > 0$ and any $\epsilon > 0$ we have

$$\mathcal{L}^n(\{\mathbf{y} \in \mathbf{B_b}(R) : \mathbf{dist}\,(\mathbf{y}, \mathbf{b} + S) \le \epsilon\}) \le 2^n R^r \epsilon^{n-r}.$$

Proof. Translating by $-\mathbf{b}$ if necessary we may assume that $\mathbf{b} = \mathbf{0}$. Rotating S if necessary we may assume that $S = \mathbf{R}^r \times \{\mathbf{0}\}$ where we identify $\mathbf{R}^n$ with $\mathbf{R}^r \times \mathbf{R}^{n-r}$ and where we make use of the fact that

$$\mathcal{L}^n(L[A]) = |\mathbf{det}\, L| \mathcal{L}^n(A) \quad \text{for any Lebesgue measurable subset of } \mathbf{R}^n.$$

Finally, we observe that

$$\begin{aligned}
&\mathcal{L}^n(\{(\mathbf{y}, \mathbf{z}) \in \mathbf{R}^r \times \mathbf{R}^{n-r} : |(\mathbf{y}, \mathbf{z})| \le R \text{ and } |\mathbf{z}| \le \epsilon\}) \\
&\qquad \le \mathcal{L}^n(\{(\mathbf{y}, \mathbf{z}) \in \mathbf{R}^r \times \mathbf{R}^{n-r} : |\mathbf{y}| \le R \text{ and } |\mathbf{z}| \le \epsilon\}) \\
&\qquad \le 2^n R^r \epsilon^{n-r}
\end{aligned}$$

by Tonelli's Theorem. □

Corollary. Suppose

(1) U is an open subset of $\mathbf{R}^n$ and $f : U \to \mathbf{R}^n$ is continuously differentiable;
(2) $\mathbf{a} \in A \subset U$ and A is convex;
(3) $M = \sup\{||\partial f(\mathbf{x})|| : x \in A\} < \infty$;
(4) $\epsilon = \sup\{||\partial f(\mathbf{x}) - \partial f(\mathbf{a})|| : x \in A\} < \infty$; and
(5) $r = \mathbf{dim}\,\mathbf{rng}\,\partial f(\mathbf{a})$. Then
(6) $\mathcal{L}^n(\mathbf{cl}\, f[A]) = 2^n(\mathbf{diam}\, A)^n M^r \epsilon^{n-r}$.

Proof. Let S be the range of $\partial f(\mathbf{a})$ and let $\mathbf{b} = f(\mathbf{a})$. We will show that

$$f[A] \subset \{\mathbf{y} \in \mathbf{B_b}(M\mathbf{diam}\, A) : \mathbf{dist}\,(\mathbf{y}, \mathbf{b} + S) \le \epsilon\mathbf{diam}\, A\}$$

and invoke the previous Proposition. By the Fundamental Theorem of Calculus we have

$$f(\mathbf{x}) = \mathbf{b} + \int_0^1 \partial f(\mathbf{a})(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))(\mathbf{x} - \mathbf{a})\, dt \tag{7}$$

for any $\mathbf{x} \in A$ from which it follows that

$$f[A] \subset \mathbf{B_b}(M\mathbf{diam}\, A).$$

We may rewrite (6) as

$$f(\mathbf{x}) = \mathbf{b} + \partial f(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \int_0^1 [\partial f(\mathbf{a})(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) - \partial f(\mathbf{a})(\mathbf{a})](\mathbf{x} - \mathbf{a})\, dt$$

for any $\mathbf{x} \in A$; it follows that

$$|(f(\mathbf{x}) - \mathbf{b}) \bullet \mathbf{w}| \le \epsilon\mathbf{diam}\, A|\mathbf{w}|$$

for any $\mathbf{x} \in A$ and any $\mathbf{w} \in S^\perp$ which implies that the distance from $f(\mathbf{x})$ to S does not exceed $\epsilon\mathbf{diam}\, A$. □

Standard Cubical Subdivision. Let

$$C = \{\mathbf{x} \in \mathbf{R}^n : 0 \le x_j < 1 \text{ for } j = 1, \ldots, n\}.$$

For each integer m let

$$\mathcal{C}_m = \{2^{-m}(\mathbf{z} + C) : \mathbf{z} \in \mathbf{Z}^n\}.$$

Note that

$$\mathbf{diam}\, C \;=\; \sqrt{n}2^{-m} \quad \text{whenever } C \in \mathcal{C}_m.$$

Theorem. Suppose U is an open subset of $\mathbf{R}^n$,

$$f : U \to \mathbf{R}^n$$

continuously differentiable and

$$A \;=\; \{\mathbf{x} \in U : \mathbf{dim}\,\mathbf{rng}\,\partial f(\mathbf{x}) < n\}.$$

Then

$$\mathcal{L}^n(f[A]) \;=\; 0.$$

Proof. For each $k = 1, 2, 3, \ldots$ let

$$A_k \;=\; \{\mathbf{x} \in U \cap \mathbf{B}_k(\mathbf{0}) : \mathbf{dim}\,\mathbf{rng}\,\partial f(\mathbf{x}) < n \text{ and } \mathbf{dist}\,(\mathbf{x}, \mathbf{R}^n \sim A) \geq \frac{1}{k}\}$$

and note that A_k is compact. Since $A = \bigcup_{k=1}^{\infty} A_k$ it will suffice to show that $\mathcal{L}^n(f[A_k]) \;=\; 0$ for each k.

So let k be a positive integer. Choose an open set G such that $A_k \subset G$ and $\mathbf{cl}\, G$ is a compact subset of U. Then $M \;=\; \max\{||\partial f(\mathbf{x})|| : \mathbf{x} \in \mathbf{cl}\, G\} < \infty$.

Suppose $0 < \epsilon \leq M$. Let $\mathcal{U}$ be the family of those open balls $\mathbf{U}_{\mathbf{a}}(s)$ which are subsets of G and are such that $\mathbf{a} \in A_k$ and

$$||\partial f(\mathbf{x}) - \partial f(\mathbf{a})|| \;\leq\; \epsilon \quad \text{whenever } \mathbf{x} \in \mathbf{U}_{\mathbf{a}}(s);$$

Evidently, $\mathcal{U}$ is a open covering of A_k; let ρ be its Lebesgue number. Next choose a positive integer N such that $\sqrt{n}2^{-M} < \rho$. Let $\mathcal{D}$ be the family of those $C \in \mathcal{C}_N$ such that $C \cap A_k \neq \emptyset$. By the preceding Corollary,

$$\mathcal{L}^n(f[C]) \;\leq\; 2^n(\mathbf{diam}\, C)^n M^{n-1}\epsilon \;=\; (2\sqrt{n})^n M^{n-1}\epsilon\mathcal{L}^n(C)$$

for any $C \in \mathcal{D}$. Thus

$$\begin{aligned}
\mathcal{L}^n(f[A_k]) \;&\leq\; \sum_{C \in \mathcal{D}} \mathcal{L}^n(f[C]) \\
&\leq\; \sum_{C \in \mathcal{D}} (2\sqrt{n})^n M^{n-1}\epsilon\mathcal{L}^n(C) \\
&=\; (2\sqrt{n})^n M^{n-1}\epsilon\mathcal{L}^n(\bigcup \mathcal{D}) \\
&\leq\; (2\sqrt{n})^n M^{n-1}\epsilon\mathcal{L}^n(G).
\end{aligned}$$

Owing to the arbitrariness of ϵ we may conclude that $\mathcal{L}^n(f[A_k]) = 0$.