1. Uniform Convergence.

Suppose $X$ is a set and $(Y, \sigma)$ is a metric space. We let

$$\mathcal{B}(X, Y)$$

be the set of bounded functions from $X$ to $Y$; that is, $f \in \mathcal{B}(X, Y)$ if $f : X \to Y$ and $\text{diam} \, \text{rng} \, f < \infty$. For each $f, g \in \mathcal{B}(X, Y)$ we set

$$\Sigma(f, g) = \sup \{ \sigma(f(x), g(x)) : x \in X \}.$$

**Proposition 1.1.** $\Sigma$ is metric on $\mathcal{B}(X, Y)$.

**Proof.** Suppose $f, g, h \in \mathcal{B}(X, Y)$ and $a \in X$. Then

$$\sigma(f(x), g(x)) \leq \sigma(f(x), f(a)) + \sigma(f(a), g(a)) + \sigma(g(a), g(x))$$

$$\leq \text{diam} \, \text{rng} \, f + \sigma(f(a), g(a)) + \text{diam} \, \text{rng} \, g$$

for any $x \in X$. Thus $\Sigma(f, g) < \infty$. It is evident that $\Sigma(g, f) = \Sigma(f, g)$ and that if $\Sigma(f, g) = 0$ then $f = g$.

Suppose $f, g, h \in \mathcal{B}(X, Y)$. Then

$$\sigma(f(x), h(x)) \leq \sigma(f(x), g(x)) + \sigma(g(x), h(x)) \leq \Sigma(f, g) + \Sigma(g, h)$$

for any $x \in X$ from which we conclude that $\Sigma(f, g) \leq \Sigma(f, g) + \Sigma(g, h)$. \hfill $\square$

**Example 1.1.** Suppose $Y$ is a vector space normed by $|\cdot|$ and $\sigma$ is the corresponding metric. Note that

$$\mathcal{B}(X, Y)$$

is then the set of functions $f : X \to Y$ such that

$$\sup \{ |f(x)| : x \in A \} < \infty.$$

We set

$$||f|| = \sup \{ |f(x)| : x \in X \}$$

whenever $f \in \mathcal{B}(X, Y)$

and note that

$$\Sigma(f, g) = ||f - g||$$

whenever $f, g \in \mathcal{B}(A, Y)$.

Obviously,

$$||f|| = 0 \iff f = 0 \quad \text{whenever} \quad f \in \mathcal{B}(X, Y).$$

If $c \in \mathbb{R}$ and $f \in \mathcal{B}(X, Y)$ we have

$$||cf|| = \{ |(cf)(x)| : x \in X \} = \{ |c||f(x)| : x \in X \} = |c||\{ |f(x)| : c \in X \} = |c||f||.$$

Moreover,

$$||f + g|| = \sup \{ |f(x) + g(x)| : x \in X \} \leq ||f|| + ||g||$$

whenever $f, g \in \mathcal{B}(X, Y)$. In particular, $\mathcal{B}(X, Y)$ is a linear subspace of $Y^X$. Thus $\mathcal{B}(X, Y)$ is a normed vector space with respect to $\| \cdot \|$.

**Proposition 1.2.** $\Sigma$ is complete if $\sigma$ is complete.

**Proof.** Suppose $f$ is a Cauchy sequence in $\mathcal{B}(X, Y)$ with respect to $\Sigma$. Then, for each $x \in X$, $\mathbb{N} \ni \nu \mapsto f_\nu(x)$ is a Cauchy sequence in $Y$ and so, as $\sigma$ is complete, converges to some $g(x)$.

We now show that $\lim_{\nu \to \infty} \Sigma(f_\nu, g) = 0$. Suppose $0 < \eta < \epsilon < \infty$. Let $N \in \mathbb{N}$ be such that

$$\mu, \nu \in \mathbb{N} \text{ and } \mu, \nu \geq N \implies \Sigma(f_\mu, f_\nu) < \eta.$$
Then for any \( x \in X \) and any \( \mu, \nu \in \mathbb{N} \) with \( \mu, \nu \geq N \) we have
\[
\sigma(f_\mu(x), g(x)) \leq \sigma(f_\mu(x), f_\nu(x)) + \sigma(f_\nu(x), g(x)) \\
\leq \Sigma(f_\mu, f_\nu) + \sigma(f_\nu(x), g(x)) \\
< \eta + \sigma(f_\nu(x), g(x)).
\]
Letting \( \nu \to \infty \) we find that if \( \mu \geq N \) then
\[
\sigma(f_\mu(x), g(x)) \leq \eta \quad \text{for any } x \in X
\]
which implies \( \Sigma(f_\mu, g) \leq \eta < \epsilon \). That is, \( f_\nu \to g \) as \( \nu \to \infty \) with respect to \( \Sigma \), as desired. \( \square \)

Now suppose \( X \) is a topological space. Let
\[
\mathcal{C}(X, Y) = \{ f \in \mathcal{B}(X, Y) : f \text{ is continuous} \}.
\]

**Theorem 1.1.** \( \mathcal{C}(X, Y) \) is a closed subset of \( \mathcal{B}(X, Y) \).

**Remark 1.1.** It follows that \( \mathcal{C}(X, Y) \) is complete with respect to the metric on it induced by \( \Sigma \) provided \( Y \) is complete.

**Proof.** Suppose \( g \in \text{cl}\mathcal{C}(X, Y) \).

Suppose \( a \in X \) and let \( \epsilon > 0 \). Since \( g \in \text{cl}\mathcal{C}(X, Y) \) we there is \( f \in U(g, \epsilon/3) \cap F \).

Since \( f \) is continuous at \( a \) there is an open subset \( U \) of \( X \) such that
\[
x \in U \Rightarrow \sigma(f(x), f(a)) \leq \epsilon/3.
\]
Then
\[
\sigma(g(x), g(a)) \leq \sigma(g(x), f(x)) + \sigma(f(x), f(a)) + \sigma(f(a), g(a)) \\
\leq \Sigma(f, g) + \epsilon/3 + \Sigma(f, g) \\
\leq \epsilon.
\]
So \( g \) is continuous and, therefore, \( \mathcal{C}(X, Y) \) is a closed subset of \( \mathcal{B}(X, Y) \). \( \square \)

**Remark 1.2.** Suppose \( X \) is compact and let
\[
\mathcal{K}(X, Y) = \{ f \in \mathcal{Y}^X : f \text{ is continuous} \}.
\]

Then
\[
\mathcal{K}(X, Y) \subset \mathcal{B}(X, Y).
\]
If \( Y \) is complete then \( \mathcal{K}(X, Y) \) is also complete by virtue of the preceding Theorem. In particular, if \( Y \) is a Banach space so is \( \mathcal{K}(X, Y) \).

**Remark 1.3.** For each \( \nu \in \mathbb{N} \) let \( f_\nu(x) = x^\nu \), \( 0 \leq x \leq 1 \). Evidently,
\[
\lim_{\nu \to \infty} f_\nu(x) = \begin{cases} 
0 & \text{if } 0 \leq x < 1, \\
1 & \text{if } x = 1.
\end{cases}
\]
Thus the pointwise limit is not continuous and, therefore, the convergence is not uniform. Indeed, if \( \mu, \nu \in \mathbb{N} \) and \( \nu > \mu \) then
\[
(f_\mu - f_\nu)(x) = x^\mu(1 - x^{\nu-\mu}) \to 1 \quad \text{as } n \to \infty
\]
which implies that
\[
\lim_{\nu \to \infty} ||f_\mu - f_\nu|| = 1 \quad \text{for any } \mu \in \mathbb{N}.
\]