1. The Theorems of Fubini and Tonelli.

Suppose $n, m \in \mathbb{N}^+$ and $0 < m < n$. We identify $\mathbb{R}^n$ with $\mathbb{R}^m \times \mathbb{R}^{n-m}$ in the natural way.

For each function f with domain $\mathbb{R}^n$ and each $x \in \mathbb{R}^m$ we let

$$\mathbf{s}_x(f) = \{(y, f(x,y)) : y \in \mathbb{R}^{n-m}\};$$

thus $\mathbf{s}_x(f)$ is a function with domain $\mathbb{R}^{n-m}$ such that

$$\mathbf{s}_x(f)(y) = f(x,y) \quad \text{whenever } y \in \mathbb{R}^{n-m}.$$

Lemma 1.1. Suppose $f \in \mathcal{F}_n^+$ and

$$F(x) = \mathbf{l}_{n-m}(\mathbf{s}_x(f) \quad \text{for } x \in \mathbb{R}^m.$$

Then

$$\mathbf{l}_m(F) \le \mathbf{l}_n(f).$$

Proof. Suppose $s \in S^+_{n,\uparrow}$ and $f \le \sup s$. Let S be the sequence in $\mathcal{F}_m^+$ such that, for each $\nu \in \mathbb{N}$, $S_\nu(x) = I^+_{n-m}(\mathbf{s}_x(s_\nu))$ for $x \in \mathbb{R}^m$. Then $S \in S^+_{n-m,\uparrow}$ and $F \le \sup S$. It follows that

$$\mathbf{l}_m(F) \le I^+_{m,\uparrow} S = I^+_{n,\uparrow}(s);$$

the Lemma follows. □

Theorem 1.1. Fubini's Theorem. Suppose $f \in \mathbf{Leb}_n$,

$$X = \{x \in \mathbb{R}^m : \mathbf{s}_x(f) \in \mathbf{Leb}_{n-m}\},$$

and

$$F : \mathbb{R}^m \to \mathbb{R}$$

is such that

$$F(x) = \mathbf{L}_{n-m}(\mathbf{s}_x(f)) \quad \text{whenever } x \in X.$$

Then $\mathcal{L}^m(\mathbb{R}^m \sim X) = 0$, $F \in \mathbf{Leb}_m$ and

$$\mathbf{L}_n(f) = \mathbf{L}_m(F).$$

Proof. **Part One** Choose sequences ϵ and η in $(0, \infty)$ such that $\lim_{\nu\to\infty} \epsilon_\nu = 0$ and $\sum_{\nu=1}^\infty \eta_\nu < \infty$.

Next, choose a sequence s in $\mathcal{S}_n$ such that $\mathbf{l}_n(|f - s_\nu|) \le \epsilon_\nu \eta_\nu$ for $\nu \in \mathbb{N}$.

For each $\nu \in \mathbb{N}$ let $S_\nu \in \S m$ be such that

$$S_\nu(x) = \mathbf{L}_{n-m}(\mathbf{s}_x(s_\nu)) \quad \text{for } x \in \mathbb{R}^m.$$

For each $\nu \in \mathbb{N}$ let

$$j_\nu(x) = \mathbf{l}_{n-m}(|\mathbf{s}_x(f) - \mathbf{s}_x(s_\nu)|) \quad \text{for } x \in \mathbb{R}^m;$$

thus $j_\nu \in \mathcal{F}_m^+$.

For each $\nu \in \mathbb{N}$ let

$$E_\nu = \{x \in \mathbb{R}^m : j_\nu(x) \le \epsilon_\nu\}.$$

let

$$D = \cup_{N=0}^\infty \cap_{\nu=N}^\infty E_\nu.$$

Part Two $D \subset X$.

Suppose $x \in D$. Choose a positive integer N such that $x \in \bigcap_{\nu=N}^\infty E_\nu$. Then for any $\nu \in \mathbb{N}$ with $\nu \ge N$ we have $\mathbf{l}_{n-m}(|\mathbf{s}_x(f) - \mathbf{s}_x(s_\nu)|) = j_\nu(x) \le \epsilon_\nu$ which implies $\mathbf{s}_x(f) \in \mathbf{Leb}_{n-m}$.

Part Three $\mathcal{L}^n(\mathbb{R}^n \sim X) = 0$.

For each $\nu \in \mathbb{N}$ we use the Lemma to estimate

$$\mathcal{L}^n(\mathbb{R}^m \sim E_\nu) = \mathbf{l}_m\left(1_{\mathbb{R}^m \sim E_\nu}\right) \leq \frac{1}{\epsilon_\nu}\mathbf{l}_{n-m}(j_\nu) \leq \frac{1}{\epsilon_\nu}\mathbf{l}_n(|f - s_\nu|) \leq \eta_\nu;$$

Consequently,

$$\begin{aligned} \mathcal{L}^m(\mathbb{R}^n \sim D) &= \mathcal{L}^m\left(\bigcap_{N=0}^{\infty}\bigcup_{\nu=N}^{\infty}\mathbb{R}^m \sim E_\nu\right) \\ &\leq \inf_N \ \mathcal{L}^m\left(\bigcup_{\nu=N}^{\infty}\mathbb{R}^m \sim E_\nu\right) \\ &\leq \inf_N \ \sum_{\nu=N}^{\infty}\mathcal{L}^m(\mathbb{R}^m \sim E_\nu) \\ &\leq \inf_N \ \sum_{\nu=N}^{\infty}\eta_\nu \\ &= 0. \end{aligned}$$

Since $D \subset X$ we have $\mathbb{R}^m \sim X \subset \mathbb{R}^m \sim D$ so $\mathcal{L}^m(\mathbb{R}^m \sim X) = 0$.

Part Four. Suppose $\nu \in \mathbb{N}$. For $x \in X$ we have

$$\begin{aligned} |F(x) - S_\nu(x)| &= |\mathbf{L}_{n-m}(\mathbf{s}_x(f)) - \mathbf{L}_{n-m}(\mathbf{s}_x(s_\nu))| \\ &= |\mathbf{L}_{n-m}(\mathbf{s}_x(f)) - \mathbf{s}_x(s_\nu))| \\ &\leq \mathbf{l}_{n-m}(|\mathbf{s}_x(f) - \mathbf{s}_x(s_\nu)|) \\ &\leq \mathbf{l}_{n-m}(|\mathbf{s}_x(f - s_\nu)|); \end{aligned} \tag{1}$$

from the Lemma we infer that

$$\mathbf{l}_m(|F - S_\nu|) = \mathbf{l}_m(1_X|F - S_\nu|) \leq \mathbf{l}_n(|f - s_\nu|).$$

Owing to the arbitrariness of ν we conclude that $F \in \mathbf{Leb}_m$.

Part Five. Suppose $\nu \in \mathbb{N}$. Since we have

$$\begin{aligned} |\mathbf{L}_m(F) - \mathbf{L}_n(f)| &\leq |\mathbf{L}_m(F) - \mathbf{L}_m(S_\nu)||\mathbf{L}_n(f) - \mathbf{L}_n(s_\nu)| \\ &= |\mathbf{L}_m(F - S_\nu)| + |\mathbf{L}_n(f - s_\nu)| \\ &\leq \mathbf{l}_m(|F - S_\nu|) + \mathbf{l}_m(|f - s_\nu|) \\ &\leq 2\mathbf{l}_m(|f - s_\nu|); \end{aligned}$$

letting $\nu \to \infty$ we infer that $\mathbf{L}_m(F) = \mathbf{L}_n(f)$, as desired. □

Theorem 1.2. Tonelli's Theorem. Suppose $f \in \mathbf{Leb}_n^+$,

$$X = \{x \in \mathbb{R}^m : \mathbf{s}_x(f) \in \mathbf{Leb}_{n-m}^+\},$$

and

$$F : \mathbb{R}^m \to [0, \infty]$$

is such that

$$F(x) = \mathbf{l}_{n-m}(\mathbf{s}_x(f)) \quad \text{whenever } x \in X.$$

Then $\mathcal{L}^m(\mathbb{R}^m \sim X) = 0$, $F \in \mathbf{Leb}_m^+$ and

$$\mathbf{l}_n(f) = \mathbf{l}_m(F).$$

Proof. Choose a nondecreasing sequence g in $\mathcal{F}_n^+ \cap \mathbf{Leb}_n$ such that $f = \sup_\nu g_\nu$. For each $\nu \in \mathbb{N}$ let

$$X_\nu = \{x \in \mathbb{R}^m : \mathbf{s}_x(g_\nu) \in \mathbf{Leb}_{n-m}\}.$$

Then

$$X \subset \cap_{\nu=0}^{\infty} X_\nu$$

so $\mathcal{L}^m(\mathbb{R}^m \sim X) = 0$ by Fubini's Theorem. The remaining assertions follow by applying combining Fubini's Theorem with f there replaced by g_ν, $\nu \in \mathbb{N}$ and invoking the Monotone Convergence Theorem. □