Tangency.

Let $X$ be a normed vector space.

**Definition.** Suppose $v \in X$ and $C \subset X$. We say $C$ is a cone with vertex $v$ if

$$x \in C \sim \{v\} \text{ and } t \geq 0 \Rightarrow v + t(x - v) \in C.$$ 

Note that the empty set is a cone with vertex $v$ and that $v \in C$ if $C \sim \{v\} \neq \emptyset$.

**Proposition.** Suppose $v \in X$ and $C$ is a nonempty family of cones with vertex $v$. Then $\bigcup C$ is a cone with vertex $v$.

**Proof.** This is immediate. □

**Proposition.** Suppose $v \in X$ and $C$ is a cone with vertex $v$. Then the closure of $C$ is a cone with vertex $v$.

**Proof.** Exercise. □

**Definition.** Suppose $A \subset X$, $a \in \text{acc } A$. For each $\delta > 0$ we let

$$\text{Tan}_a(A, \delta) = \text{cl} \{t(x - a) : t \geq 0, \text{ and } x \in (A \sim \{a\}) \cap B_a(\delta)\}.$$ 

Note that, by virtue of the previous Proposition, $\text{Tan}_a(A, \delta)$ is a closed cone with vertex 0.

We let

$$\text{Tan}_a(A) = \bigcap_{\delta > 0} \text{Tan}_a(A, \delta)$$

and we let

$$\text{Nor}_a(A) = \{\omega \in X^* : \omega(v) \leq 0 \text{ whenever } v \in \text{Tan}_a(A)\}.$$ 

Note that $\text{Tan}_a(A)$ and $\text{Nor}_a(A)$ are closed cones in $X$ and $X^*$, respectively, by virtue of the first Proposition above.

In case $X$ is an inner product space we will also let

$$\text{Nor}_a(A) = \{w \in X : v \cdot w \leq 0 \text{ whenever } v \in \text{Tan}_a(A)\}$$

and rely on the context to resolve the ambiguity.

**Theorem.** Suppose $X$ is finite dimensional, $A \subset X$, $a \in \text{acc } A$. Then $\text{Tan}_a(A) \neq \emptyset$. Moreover, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{cl } A \cap B_a(\delta) \subset a + \{v \in X : \text{dist}(v, \text{Tan}_a(A)) \leq \epsilon|v|\}$$

**Proof.** Let $K = \{u \in X : |u| = 1\}$ and note that $K$ is compact because $X$ is finite dimensional. Let $L = K \cap \text{Tan}_a(A)$ and, for each $\delta > 0$, let $T_\delta = K \cap \text{Tan}_a(A, \delta)$. Then $\{T_\delta : \delta > 0\}$ is a nonempty nested family of closed subsets of the compact set $K$ whose nonempty intersection is $L$. Moreover, if $U$ is an open set containing $L$ then there is $\delta > 0$ such that $T_\delta \subset U$.

Now suppose $\epsilon > 0$. Let

$$U = \{v \in X \sim \{0\} : \text{dist}(v, \text{Tan}_a(A)) < \epsilon|v|\}$$

and note that $U$ is open. Since $L \subset U$ and $U$ is open there is $\delta > 0$ such that $T_\delta \subset U$. □

**Proposition.** Suppose $A \subset X$, $a \in \text{acc } A$ and $v \in X \sim \{0\}$. The following are equivalent.

(i) $v \in \text{Tan}_a(A)$. 

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(ii) For each \( \epsilon > 0 \) and \( \delta > 0 \) there are \( s > 0 \) and \( x \in (A \sim \{a\}) \cap B_\delta(0) \) such that
\[
|(x-a) - sv| \leq \epsilon|x-a|.
\]

**Proof.** Suppose \( v \in \Tan_a(A) \), \( \epsilon > 0 \) and \( \delta > 0 \). Let \( \eta \) be such that \( 0 < \eta < 1 \) and \( \frac{1}{1 - \eta} \leq \epsilon \). Since \( v \) is a member of the closure of \( \Tan_a(A, \delta) \), there exist \( x \in (A \sim \{a\}) \cap B_\delta(0) \) and \( t \geq 0 \) such that \( |t(x-a) - v| \leq \eta|v| \). This implies \( |t(x-a) - v| \leq \eta|v| \) so that \( t|x-a| \leq (1-\eta)|v| \). In particular, \( t|x-a| > 0 \). Let \( s = \frac{1}{t} \). Then
\[
|(x-a) - sv| = \frac{1}{t} |t(x-a) - v| \leq \frac{|x-a|}{1-\eta} |\eta| |v| \leq \epsilon|x-a|
\]
so (ii) holds.

On the other hand, suppose (ii) holds, let \( \delta > 0 \) and let \( \rho > 0 \). Let \( \zeta \) be such that \( 0 < \zeta < 1 \) and \( \frac{|v|}{1-\zeta} \leq \rho \). Let \( s > 0 \) and \( x \in (A \sim \{a\}) \cap B_\delta(0) \) such that \( |(x-a) - sv| \leq \zeta|x-a| \). Then \( |x-a| - s|v| \leq \zeta|x-a| \) so \( s|v| \geq (1-\zeta)|x-a| \). Set \( t = \frac{1}{s} \). Then
\[
|t(x-a) - v| = \frac{1}{s} |(x-a) - sv| \leq \frac{|v|}{(1-\zeta)|x-a|} |\zeta| |x-a| \leq \rho.
\]

Owing to the arbitrariness of \( \rho \) we infer that \( v \in \Tan_a(A, \delta) \). Owing to the arbitrariness of \( \delta \) we infer that (i) holds.

**Theorem.** Suppose \( X \) and \( Y \) are normed spaces, \( A \subset X \), \( a \in \text{int} \ A \), \( f : A \rightarrow Y \) and \( f \) is differentiable at \( a \). Then
\[
\text{range} \partial f(a) \sim \{0\} \subset \Tan_{f(a)}(f[A]).
\]

**Proof.** Suppose \( v \in X \) and \( w = \partial f(a)(v) \neq 0 \). Let \( \epsilon > 0 \) and choose \( \eta \) such that \( 0 < |v|\eta < |w| \) and \( \frac{|v|}{|w| - |v|\eta} \leq \epsilon |v| \). Choose \( \delta > 0 \) such that
\[
x \in A \cap B_\delta(0) \Rightarrow |f(x) - f(a) - \partial f(a)(x-a)| \leq \eta|x-a|.
\]
If \( t > 0 \) and \( t|v| \leq \delta \) we have \( |f(a+tv) - f(a) - tw| \leq \eta|v| \) so \( |f(a+tv) - f(a)| \geq t(|v| - \eta|v|) \). Consequently,
\[
|f(a+tv) - f(a) - tw| \leq \frac{t\eta|v|}{w}|f(a+tv) - f(a)| \leq \frac{\eta|v|}{|w| - \eta|v|} |f(a+tv) - f(a)| \leq \epsilon |f(a+tv) - f(a)|.
\]
The Theorem now follows from a previous Proposition. \( \square \)

**Theorem.** Suppose \( X \) and \( Y \) are normed spaces, \( X \) is finite dimensional, \( A \) is an open subset of \( X \), \( f \) is differentiable at each point of \( A \) and \( b \in \text{range} \ f \).

Suppose, additionally, that

(i) \( \ker \partial f(a) = \{0\} \) whenever \( a \in A \) and \( f(a) = b \);

(ii) there is \( s > 0 \) such that \( f^{-1}[B_s(b)] \) is a compact subset of \( A \).

Then \( b \in \text{acc} \text{range} \ f \), \( \{a \in A : f(a) = b\} \) is finite and

\[
(1) \quad \Tan_b(\text{range} \ f) = \bigcup \{\text{range} \partial f(a) : a \in A \text{ and } f(a) = b\}.
\]

**Proof.** We have already shown that the right hand side of (1) is a subset of the left hand side. So suppose \( w \in \Tan_b(\text{range} \ f) \), \( |w| = 1 \) and \( \epsilon > 0 \). We will obtain \( a \in A \) and \( v \in X \) such that \( f(a) = b \) and \( |w - \partial f(a)(v)| \leq \epsilon \). This will show that \( w \) is a point of the closure of the range of \( \partial f(a) \). Since \( X \) is finite dimensional, the range of \( \partial f(a) \) is closed so the proof will be complete.
Let \( K = \{ a \in A : f(a) = b \} \). \( K \) is closed relative to \( A \) because \( f \) is continuous. Since \( K \) is a subset of the compact set \( f^{-1}[B_b(s)] \) we infer that \( K \) is compact. For each \( a \in K \) choose \( m_a, M_a \) such that \( 0 < m_a \leq M_a < \infty \) and

\[
m_a |v| < |\partial f(a)(v)| < M_a |v| \quad \text{whenever} \ v \in X \sim \{0\};
\]

this is possible because \( X \) is finite dimensional and \( \ker \partial f(a) = \{0\} \). For any \( a, x \in A \) we have

\[
|f(x) - f(a)| - |\partial f(a)(x - a)| \leq |f(x) - f(a) - \partial f(a)(x - a)|;
\]

it follows that for each \( a \in K \) there is \( \rho_a > 0 \) such that \( B_a(\rho_a) \subset X \) and

\[
m_a |x - a| \leq |f(x) - f(a)| \leq M_a |x - a| \quad \text{whenever} \ x \in B_{\rho_a}(a).
\]

In particular, \( f(x) \neq f(a) \) for any \( a \in K \) and any \( x \in B_a(\rho_a) \). As \( K \) is compact, we infer that that \( K \) is finite. Let \( \rho > 0 \) be such that \( \rho < \rho_a \) for \( a \in K \) and

\[
(2) \quad \frac{1}{m_a} \frac{|f(x) - f(a) - \partial f(a)(x - a)|}{|x - a|} \leq \frac{\epsilon}{2} \quad \text{whenever} \ x \in B_a(\rho).
\]

Let \( F_\sigma = f^{-1}[B_{\sigma}(b)] \) for \( 0 < \sigma \leq s \) and note that \( F_\sigma \) is closed relative to \( A \) because \( f \) is continuous. Now \( \{ F_\sigma : 0 < \sigma \leq s \} \) is a nested family of closed subsets of the compact set \( F_s \) with intersection \( K \). It follows that there is \( \sigma \) such that \( 0 < \sigma \leq s \) and \( F_\sigma \subset \cup \{ B_a(\rho) : a \in A \} \). Since \( w \in \text{Tan}_b(\text{rng} f) \) we may choose \( y \in \text{rng} f \cap (F_\sigma \sim \{b\}) \) such that

\[
|\frac{1}{|y - b|}(y - b) - w| \leq \frac{\epsilon}{2}.
\]

Let \( a \in A \) and \( x \in B_b(\rho_a) \) be such that \( y = f(x) \). Then

\[
|w - \partial f(a)(\frac{1}{|y - b|}(x - a))| = |w - \frac{1}{|y - b|}(y - b) + \frac{1}{|f(x) - f(a)|}(f(x) - f(a) - \partial f(a)(x - a))|
\]

\[
\leq |w - \frac{1}{|y - b|}(y - b)| + \frac{|f(x) - f(a) - \partial f(a)(x - a)|}{|x - a|} \frac{|x - a|}{|f(x) - f(a)|} \leq \epsilon.
\]

\( \square \)

**Theorem.** Suppose \( X \) and \( Y \) are finite dimensional normed spaces, \( A \subset X, a \in \text{int} A, \)

\[
f : A \rightarrow Y
\]

and \( f \) is continuous at \( a \). Then \( f \) is differentiable at \( a \) if and only if

\[
\text{Tan}_{(a,f(a))}(f)
\]

is a linear function from \( X \) to \( Y \) in which case

\[
\text{Tan}_{(a,f(a))}(f) = \partial f(a).
\]

**Proof.** Suppose \( f \) is differentiable at \( a \). Let \( F(x) = (x, f(x)) \) for \( x \in A \); note that \( F \) is differentiable at \( a \) and that \( \partial F(a)(v) = (v, \partial f(a)) \) whenever \( v \in X \). We may apply the previous Theorem with \( b \) and \( f \) there replaced by \( (a, b) \) and \( F \), respectively, to deduce that \( \text{Tan}_{(a,f(a))}(f) = \partial f(a) \).

On the other hand, suppose that \( L = \text{Tan}_{(a,f(a))}(f) \) is a linear function from \( X \) to \( Y \). Keeping in mind that all norms on a finite dimensional vector space are equivalent, we may suppose \( ||(x, y)|| = |x| + |y| \) for \( (x, y) \in X \times Y \). We may suppose without loss of generality that \( a = 0 \) and \( f(a) = 0 \).
Let $\epsilon > 0$ and choose $\eta > 0$ such that $\eta(1 + ||L||) < 1$, $\frac{1 + ||L||}{1 - \frac{1}{1 + ||L||}} \eta \leq 2$ and 

$$(1 + ||L||)3\eta \leq \epsilon.$$

Choose $\zeta > 0$ such that if $(x, y) \in f \cap B_0(\zeta)$ then

$$\text{dist}((x, y), L) < \eta|(x, y)|.$$

Finally, using the fact that $f$ is continuous at 0, choose $\delta > 0$ such that if $x \in B_0(\delta)$ then $x \in A$ and $|(x, f(x))| \leq \zeta$.

Suppose $x \in B_0(\delta)$ and let $y = f(x)$. Then $(x, y) \in f \cap B_0(\zeta)$ so

$$\text{dist}((x, y), L) < \eta|(x, y)|.$$

We may choose $v \in X$ such that $|(x, y) - (v, L(v))| < \eta(x, y)$ so $|x - v| + |y - L(v)| \leq |x| + |y|$. Thus

$$|y| \leq |y - L(v)| + |L(v-x)| + |L(x)| \leq (1 + ||L||)(|x - v| + |y - L(v)|) + ||L|||x| \leq (1 + ||L||)\eta(|x| + |y|) + ||L|||x|$$

so

$$(1 - (1 + ||L||)\eta)|y| \leq (1 + (1 + ||L||)\eta)|x|$$

so $|y| \leq 2|x|$. It follows that

$$|y - L(x)| \leq |y - L(v)| + ||L|||x - v| \leq (1 + ||L||)\eta(|x| + |y|) \leq (1 + ||L||)3\eta|x| \leq \epsilon|x|.$$

Thus $f$ is differentiable at $a = 0$ and its differential is $L$. □

**Theorem.** Suppose $X$ is a normed vector space, $U$ is an open subset of $X$,

$$f : U \to \mathbb{R},$$

$a \in \text{acc } A$ and $f$ is differentiable at $a$.

If $f(x) \leq f(a)$ for $x \in A$ then $\partial f(a) \in \text{Nor}_a(A)$.

If $f(x) \geq f(a)$ for $x \in A$ then $-\partial f(a) \in \text{Nor}_a(A)$.

**Proof.** Exercise. □

Now suppose $X$ is an inner product space. In this case, as we indicated before, we set

$$\text{Nor}_a(A) = \{w \in X : v \cdot w \leq 0 \text{ whenever } v \in \text{Tan}_a(A)\}.$$

Note the the polarity of the inner product carries the present normal cone to the former normal cone.

**Definition. The gradient.** Suppose $A \subset X$, $f : A \to \mathbb{R}$, and $f$ is differentiable at $a$. We let

$$\nabla f(a),$$

the gradient of $f$ at $a$, be the counter image of $\partial f(a)$ under the polarity of the inner product; that is, $\nabla f(a)$ is the unique vector in $X$ satisfying

$$\partial f(a)(v) = v \cdot \nabla f(a), \quad v \in X.$$

In this situation the conclusion of the previous Theorem becomes

If $f(x) \leq f(a)$ for $x \in A$ then $\nabla f(a) \in \text{Nor}_a(A)$.

If $f(x) \geq f(a)$ for $x \in A$ then $-\nabla f(a) \in \text{Nor}_a(A)$. 

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