1. Summation.

Let X be a set.

1.1. **Finite summation.** *The stuff in this subsection is now in* `binary.tex`

Suppose Y is a set and

$$\cdot + \cdot : Y \times Y \to Y$$

is such that

(i) $x + (y + z) = (x + y) + z$ whenever $x, y, z \in Y$;
(ii) $x + y = y + x$ whenever $x, y \in Y$;
(iii) there is $0 \in Y$ such that $y + 0 = y = 0 + y$ whenever $y \in Y$.

For example, Y could be an Abelian group or Y could be $[0, \infty]$ where $+$ on $[0, \infty) \times [0, \infty)$ is addition in the Abelian group $\mathbb{R}$ and where

$$y + \infty = \infty = \infty + y \quad \text{whenever } y \in [0, \infty].$$

Definition 1.1. For $f, g \in Y^X$ we define $f + g \in Y^X$ by letting

$$(f + g)(x) = f(x) + g(x) \quad \text{for } x \in X$$

and we note that appropriately reformulated versions of (i),(ii) and (iii) hold. We let

$$0 : X \to Y$$

be such that $0(x) = 0$ for $x \in X$.

Definition 1.2. For $f \in Y^X$ we let

$$\mathbf{spt}\, f = \{x \in X : f(x) \neq 0\}$$

and call this subset of X the **support of f.** We let

$$\left(Y^X\right)_0 = \{f \in Y^X : \mathbf{spt}\, f \text{ is finite}\}$$

and note that $\left(Y^X\right)_0$ is closed under addition.

Definition 1.3. Whenever $A \subset X$ and $f \in Y^X$ we let

$$f_A \in Y^X$$

be such that

$$f_A(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in X \sim A. \end{cases}$$

Proposition 1.1. Suppose F is a finite subset of X. There is one and only one function

$$S_F : Y^X \to Y$$

such that

(i) $S_F(0) = 0$;
(ii) $S_F(f) = S(f_{X \sim \{a\}}) + f(a)$ whenever $f \in Y^X$ and $a \in A$;
(iii) $S_F(f + g) = S_F(f) + S_F(g)$ whenever $f, g \in Y^X$.

Proof. We define S_F by induction on $|F|$ as follows. We let $S_\emptyset(0) = 0$. If $|F| > 0$ we let

$$S_F = \{(f, S_{F\sim\{a\}}(f_{X\sim\{a\}}) + f(a)) : f \in \mathcal{F}_F \text{ and } a \in F\}.$$

It is obvious that S_F is a function if $|F| = 1$. To verify that S_F is a function in case $|F| > 1$ we suppose $f \in \mathcal{F}_F$, $a, b \in F$ and $a \neq b$ and we calculate

$$\begin{aligned} S_{F\sim\{a\}}(f_{X\sim\{a\}}) + f(a) &= (S_{F\sim\{a,b\}}(f_{X\sim\{a,b\}}) + f(b)) + f(a) \\ &= S_{F\sim\{a,b\}}(f_{X\sim\{a,b\}}) + (f(b) + f(a)) \\ &= S_{F\sim\{a,b\}}(f_{X\sim\{a,b\}}) + (f(a) + f(b)) \\ &= (S_{F\sim\{a,b\}}(f_{X\sim\{a,b\}} + f(a)) + f(b) \\ &= S_{F\sim\{b\}}(f_{X\sim\{b\}}) + f(b). \end{aligned}$$

We leave to the reader the straightforward verification using induction on $|F|$ that S_F satisfies (i)-(iii). □

1.2. **Summing a function with values in** $[0, \infty]$**.** For each subset A of X let $1_A \in [0, \infty]^X$ be such that

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \sim A; \end{cases}$$

one calls 1_A the **indicator function of** A.

Note that

$$p_A = 1_A p \quad \text{whenever } p \in [0, \infty]^X.$$

Definition 1.4. For $p \in [0, \infty]^X$ we let

$$\sum p = \sup\{S_F(p) : F \subset X \text{ and } F \text{ is finite}\}.$$

Theorem 1.1. We have

(i) $\sum 0 = 0$;
(ii) $\sum 1_{\{a\}} = 1$ whenever $a \in X$;
(iii) $\sum cp = c \sum p$ whenever $0 \leq c \leq \infty$ and $p \in [0, \infty]^X$;
(iv) $\sum (p + q) = \sum p + \sum q$ whenever $p, q \in [0, \infty]^X$;
(v) $\sum p \leq \sum q$ whenever $p, q \in [0, \infty]^X$ and $p \leq q$.

Proof. (i) and (ii) are immediate. Let $\mathcal{F}$ be the family of finite subsets of X. In what follows we leave it to the reader to supply the simple proofs of the properties of S_F, $F \in \mathcal{F}$ that we shall use.

Suppose $p, q \in [0, \infty]^X$ and $0 \leq c \leq \infty$. Then

$$\sum cp = \sup_{F\in\mathcal{F}} S_F(cp) = \sup_{F\in\mathcal{F}} cS_F(p) = c \sup_{F\in\mathcal{F}} S_F(p) = c \sum p$$

so (ii) holds.

For any $F \in \mathcal{F}$ we have

$$S_F(p + q) = S_F(p) + S_F(q) \leq \sum p + \sum q$$

which implies that $\sum (p + q) \leq \sum p + \sum q$. Moreover, if $F, G \in \mathcal{F}$ we have

$$S_F(p) + S_G(q) \leq S_{F\cup G}(p) + S_{F\cup G}(q) \leq S_{F\cup G}(p + q) \leq \sum (p + q).$$

Thus (iv) holds.

Suppose $p \le q$. For any $F \in \mathcal{F}$ we have
$$S_F(p) \le S_F(q) = \sum q$$
so (v) holds. □

Suppose $p \in [0,\infty]^X$. We will sometimes write
$$\sum_A p \quad \text{or} \quad \sum_{x\in A} p(x) \quad \text{instead of} \quad \sum p_A.$$

Corollary 1.1. Suppose $p, q \in [0,\infty]^X$, $p \le q$ and $A \subset B \subset X$. Then
$$\sum_A p \le \sum_B q.$$

Proof. Note that $p_A \le q_B$. □

Example 1.1. Suppose $0 \le r < 1$. We define $p : \mathbb{N} \to [0,1)$ by letting
$$p(n) = r^n \quad \text{for } n \in \mathbb{N}.$$
Then
$$\begin{aligned}\sum_{n=0}^{\infty} r^n &= \sum p \\ &= \sup\{S_F(p) : F \text{ is a finite subset of } \mathbb{N}\} \\ &= \sup\{\sum_{n=0}^{N} r^n : N \in \mathbb{N}\} \qquad (1) \\ &= \sup\left\{\frac{1-r^{N+1}}{1-r} : N \in \mathbb{N}\right\} \\ &= \frac{1}{1-r}.\end{aligned}$$

Proposition 1.2. Suppose $\mathcal{A}$ is a partition of X and $p \in [0,\infty]^X$. Then
$$\sum p = \sum_{A\in\mathcal{A}} \sum_A p.$$

Proof. One proves by induction on $|\mathcal{A}|$ using (iv) of the preceding Theorem that the Proposition holds when $\mathcal{A}$ is finite.

Suppose $\mathcal{F}$ is a finite subfamily of $\mathcal{A}$. Then
$$\sum_{A\in\mathcal{F}} \sum_A p = \sum_{A\in\mathcal{F}} \sum p_A = \sum \sum_{A\in\mathcal{F}} p_A = \sum p_{\cup\mathcal{F}} \le \sum p.$$
Thus
$$\sum_{A\in\mathcal{A}} \sum_A p \le \sum p.$$

Suppose F is a finite subset of X. Let $\mathcal{F} = \{A \in \mathcal{A} : F \cap A \ne \emptyset\}$. Then
$$\sum p_F = \sum \sum_{A\in\mathcal{F}} p_{F\cap A} = \sum_{A\in\mathcal{F}} \sum p_{F\cap A} \le \sum_{A\in\mathcal{F}} \sum_A p \le \sum_{A\in\mathcal{A}} \sum_A p.$$
Thus
$$\sum p \le \sum_{A\in\mathcal{A}} \sum_A p.$$

□

Definition 1.5. Suppose B is a set and $p : B \to [0,\infty]^X$. (Some would say $p_b,\ b \in B$, is an indexed family of $[0,\infty]$ valued functions with domain X.) We let

$$\sum_{b\in B} p_b = \Big(X \ni x \mapsto \sum_{b\in B} p_b(x) \in [0,\infty]\Big) \in [0,\infty]^X.$$

Proposition 1.3. Suppose $p_b,\ b \in B$ is an indexed family of $[0,\infty]$ valued functions with a common domain X and $A \subset X$. Then

$$\sum_A \sum_{b\in B} p_b = \sum_{b\in B} \sum_A p_b.$$

Proof. Let $P(b,x) = p_b(x)$ for $(b,x) \in B \times X$. Apply the previous Proposition twice to P with $\mathcal{A}$ there equal $\{\{b\} \times X : b \in B\}$ and $\{B \times \{x\} : x \in X\}$. □

Proposition 1.4. Suppose $p \in [0,\infty]^X$ and $\sum p < \infty$. For each $\epsilon > 0$ there is a finite subset F of X such that

$$\sum_{X\sim A} p < \epsilon \quad \text{whenever } F \subset A \subset X.$$

Proof. Suppose $\epsilon > 0$. Let F be a finite subset of X such that

$$\sum p < \sum_F p + \epsilon.$$

Since $p = p_F + p_{X\sim F}$ we have

$$\sum_F p + \sum_{X\sim F} p = \sum p_F + \sum p_{X\sim F} = \sum p < \sum_F p + \epsilon.$$

If $F \subset A \subset X$ we have $p_{X\sim A} \le p_{X\sim F}$ so

$$\sum_{X\sim A} p \le \sum_{X\sim F} p < \epsilon.$$

□

Proposition 1.5. Suppose $p \in [0,\infty]^X$ and

$$\sum p < \infty.$$

Then $\mathbf{spt}\, p$ is countable.

Proof. Suppose n is positive integer and $A_n = \{x \in X : p(x) \ge 1/n\}$. Then $1_{A_n} \le np$ which implies $|A_n| = \sum 1_{A_n} \le \sum np = n \sum p < \infty$ so A_n is finite.

Thus $\mathbf{spt}\, p = \cup_{n=1}^{\infty} A_n$ is countable. □

1.3. **Vector valued summation.** We now assume that V is a **Banach space** which, by definition, means that V is a complete normed linear space and, under this assumption, extend the notion of summation when the support of f is *infinite*.

Definition 1.6. Suppose $f \in V^X$ and $A \subset X$. We say f is **summable over** A if
$$\sum_A |f| < \infty.$$
We say f is **summable** if f is summable over X.

If $A \subset X$ and $f \in V^X$ is summable then, as $|f_A| \leq |f|$ we find that f is summable over A.

Evidently,
$$\{f \in V^X : f \text{ if summable}\} \quad \text{is a linear subspace of } V^X.$$

Theorem 1.2. There is one and only one linear function
$$\sum : \{f \in V^X : f \text{ is summable}\} \to V$$
such that
$$\sum f_F = S_F(f) \quad \text{if } f \in V^X \text{ and } F \text{ is a finite subset of } X$$
and
$$\left|\sum f\right| \leq \sum |f| \quad \text{whenever } f \in V^X \text{ and } f \text{ is summable.}$$

Proof. Let $\rho(f) = \sum |f|$ for $f \in V^X$. Then
$$\rho(cf) = |c|\rho(f) \quad \text{whenever } c \in \mathbf{R} \text{ and } f \in V^X$$
and
$$\rho(f+g) \leq \rho(f) + \rho(g) \quad \text{whenever } f, g \in V^X.$$

By induction on $|\mathbf{spt}\, f|$ one shows that
$$|S_F(f)| \leq S_F(|f|) \leq \rho(f) \quad \text{whenever } f \in V^X \text{ and } F \text{ is a finite subset of } F.$$
The Theorem now follows by applying the Abstract Closure Principle to
$$\left(V^X\right)_0 \ni f \mapsto S_{\mathbf{spt}\, f}(f).$$
$\square$

Remark 1.1. An alternative approach to defining $\sum f$ when f is summable is as follows. For each positive integer ν let $F_\nu = \{x \in X : |f(x)| \geq 1/\nu\}$, note that F_ν is finite and set $y_\nu = \sum_{x \in F_\nu} f(x)$. Given $\epsilon > 0$ there is a positive integer N such that $\sum_{X \sim F_N} |f| < \epsilon$. Thus if μ, ν are positive integers and $\mu, \nu \geq N$ we have
$$|y_\mu - y_\nu| \leq \sum_{x \in X \sim F_N} |f(x)| < \epsilon$$
where we have used that fact that $F_N \subset F_\mu \cap F_\nu$. Thus y is a Cauchy sequence whose limit is $\sum f$.

Definition 1.7. Whenever $f \in V^X$, A is a subset of X and f_A is summable we let
$$\sum_A f = \sum f_A.$$

Remark 1.2. Note that if f is summable and $\epsilon > 0$ there is a finite subset F of X such that

$$\left| \sum f - \sum_A f \right| < \epsilon \quad \text{whenever } F \subset A \subset X.$$

Theorem 1.3. Suppose $f \in V^X$, f is summable and $\mathcal{A}$ is a partition of X.
Then

(i) $\sum_A |f| < \infty$ for each $A \in \mathcal{A}$;
(ii) $\sum_{A \in \mathcal{A}} |\sum_A f| < \infty$;

and

$$\sum_{\cup \mathcal{A}} f = \sum_{A \in \mathcal{A}} \sum_A f. \tag{1}$$

Proof. We have

$$\sum_{A \in \mathcal{A}} \sum_A |f| = \sum |f| < \infty$$

so (i) holds. We have

$$\sum_{A \in \mathcal{A}} \left| \sum_A f \right| \leq \sum_{A \in \mathcal{A}} \sum_A |f| = \sum |f| < \infty$$

and (ii) holds.

Suppose $\epsilon > 0$. Let F be a subset of $\cup \mathcal{A}$ such that

$$\sum_{(\cup \mathcal{A}) \sim F} |f| < \frac{\epsilon}{2}.$$

Let $\mathcal{F} = \{A \in \mathcal{A} : A \cap F \neq \emptyset\}$; note that $\mathcal{F}$ is finite, $F \subset \cup \mathcal{F}$ and

$$\sum_{(\cup \mathcal{A}) \sim (\cup \mathcal{F})} |f| \leq \sum_{(\mathcal{A}) \sim F} |f| < \frac{\epsilon}{2}.$$

Using the fact that $\sum \left(\sum_{i=1}^m g_i \right) = \sum_{i=1}^m \sum g_i$ whenever $g_1, \ldots, g_m$ are summable we find that

$$\sum_{\sup \mathcal{F}} = \sum f_{\cup \mathcal{F}} = \sum \left(\sum_{A \in \mathcal{F}} f_A \right) = \sum_{A \in \mathcal{F}} \left(\sum f_A \right) = \sum_{A \in \mathcal{F}} \sum_A f.$$

Thus

$$\begin{aligned} \left| \sum_{\cup \mathcal{A}} f - \sum_{A \in \mathcal{A}} \sum_A f \right| &\leq \left| \sum_{\cup \mathcal{A}} f - \sum_{\cup \mathcal{F}} f \right| + \left| \sum_{A \in \mathcal{A}} \sum_A f - \sum_{A \in \mathcal{F}} \sum_A f \right| \\ &= \left| \sum_{(\cup \mathcal{A}) \sim (\cup \mathcal{F})} f \right| + \left| \sum_{A \in \mathcal{A} \sim \mathcal{F}} \sum_A f \right| \\ &\leq \sum_{(\cup \mathcal{A}) \sim (\cup \mathcal{F})} |f| + \sum_{A \in \mathcal{A} \sim \mathcal{F}} \sum_A |f| \\ &= 2 \sum_{(\cup \mathcal{A}) \sim (\cup \mathcal{F})} |f|; \\ &< \epsilon. \end{aligned}$$

here we have used again that summation is finitely additive and the version of the current Theorem in the case of nonnegative functions. □

1.4. **The complex exponential function.** Whenever $0 < r < \infty$ we let

$$M(r) = \sup\left\{\frac{r^n}{n!} : n \in \mathbb{N}\right\}.$$

Suppose $0 < r < \infty$. Let $N(r)$ be that integer such that $N(r) \le r < N(r)+1$. Whenever $n \in \mathbb{N}$ and $n > N(r)$ we have

$$\frac{r^n}{n!} = \left(\prod_{m=N(r)+1}^{n} \frac{r}{m}\right) \frac{r^{N(r)}}{N(r)!} < \frac{r^{N(r)}}{N(r)!}$$

from which it follows that

$$M(r) \le \frac{r^{N(r)}}{N(r)!} < \infty.$$

Suppose $z \in \mathbb{C}$. Let r be such that $|z| < r < \infty$. Then

$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!} = \frac{r^n}{n!}\left(\frac{|z|}{r}\right)^n \le M(r)\left(\frac{|z|}{r}\right)^n$$

so, by Example (1),

$$\sum_{n=0}^{\infty}\left|\frac{z^n}{n!}\right| \le M(r)\sum_{n=0}^{\infty}\left(\frac{|z|}{r}\right)^n = M(r)\frac{1}{1-\frac{|z|}{r}} < \infty.$$

Thus we may define

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

(One also writes $\exp(z)$ for e^z.) I claim that

$$e^{z+w} = e^z e^w \quad \text{for } z, w \in \mathbb{C}.$$

We prove this as follows. Fix $z, w \in \mathbb{C}$. Let

$$T = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m \le n\}$$

and let

$$f(m,n) = \frac{z^m}{m!}\frac{w^{n-m}}{(n-m)!} \quad \text{for } (m,n) \in T.$$

For each $n \in \mathbb{N}$ let $A_n = \{m \in \mathbb{N} : m \le n\}$ and for each $m \in \mathbb{N}$ let $B_m = \{n \in \mathbb{N} : m \le n\}$. Note that

$$\{A_n : n \in \mathbb{N}\} \quad \text{and} \quad \{B_m : m \in \mathbb{N}\} \quad \text{are partitions of } T.$$

We have

$$\sum_{(m,n)\in T} |f(m,n)| = \sum_{n=0}^{\infty}\sum_{m\in A_n} |f(m,n)|$$

so that, using more traditional notation,

$$\begin{aligned}\sum_{(m,n)\in T} \frac{|z|^m}{m!}\frac{|w|^{n-m}}{(n-m)!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{|z|^m}{m!}\frac{|w|^{n-m}}{(n-m)!} \\ &= \sum_{n=0}^{\infty} \frac{(|z|+|w|)^n}{n!} \\ &= e^{|z|+|w|} \\ &< \infty.\end{aligned}$$

Applying the previous theorem twice we infer that

$$\sum_{n=0}^{\infty}\sum_{m\in A_n} f(m,n) = \sum_{(m,n)\in T} f(m,n) = \sum_{m=0}^{\infty}\sum_{n\in B_m} f(m,n)$$

so that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{z^m}{m!}\frac{w^{n-m}}{(n-m)!} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \frac{z^m}{m!}\frac{w^{n-m}}{(n-m)!}.$$

Thus

$$\begin{aligned}e^{z+w} &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{z^m}{m!}\frac{w^{n-m}}{(n-m)!} \\ &= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \frac{z^m}{m!}\frac{w^{n-m}}{(n-m)!} \\ &= \sum_{m=0}^{\infty} \frac{z^m}{m!} e^w \\ &= e^z e^w.\end{aligned}$$

Finally, let us show that

$$\lim_{w\to z} \frac{e^w - e^z}{w-z} = e^z. \tag{3}$$

Suppose $z, w \in \mathbb{C}$ and $0 < |w-z| < 1$. Then

$$e^{w-z} = 1 + (w-z) + (w-z)^2 \sum_{n=2}^{\infty} \frac{(w-z)^{n-2}}{n!}$$

so

$$\begin{aligned}\left|\frac{e^{w-z}-1}{w-z} - 1\right| &= |w-z| \left|\sum_{n=2}^{\infty} \frac{(w-z)^{n-2}}{n!}\right| \le |w-z| \sum_{n=2}^{\infty} |w-z|^{n-2} \\ &= |w-z| \frac{1}{1-|w-z|}.\end{aligned}$$

Since

$$\frac{e^w - e^z}{w-z} - e^z = e^z \left(\frac{e^{w-z}-1}{w-z} - 1\right)$$

(3) holds.