Stokes' Theorem.

Let n be a positive integer, let V be an open subset of $\mathbf{R}^n$ and let m be an integer such that $1 \le m \le n$.

Stokes' Theorem will follow rather directly from the definition of the integral of a differential form over a submanifold and the following Proposition.

Proposition. Suppose $\psi \in \mathcal{A}_0^{m-1}(\mathbf{U}^m)$. Then

$$\int_{\mathbf{U}^m} d\psi(t)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt \;=\; 0 \tag{1}$$

and

$$\int_{\mathbf{U}^{m,m,+}} d\psi(t)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt = (-1)^m \int_{\mathbf{U}^{m-1}} {\mathbf{i}_{m-1,m}}^{\#}\psi(s)(\mathbf{e}_1, \ldots, \mathbf{e}_{m-1})\, ds. \tag{2}$$

Proof. For each $j = 1, \ldots, m$ set $f_j = \mathbf{e}^j \wedge \omega$. We have

$$d\psi = \sum_{j=1}^{m} \mathbf{e}^j \wedge \partial_j \psi = \sum_{j=1}^{m} \partial_j f_j.$$

From Fubini's Theorem and the Fundamental Theorem of Calculus we conclude that (1) holds and that

$$\int_{\mathbf{U}^{m-1,m,+}} d\psi(t)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt = -\int_{\mathbf{U}^{m-1,m,+}} f_m(t)\, dt = -\int_{\mathbf{U}^{m-1}} f_m \circ \mathbf{i}_{m-1,m}(s)\, ds.$$

For any $t \in \mathbf{U}^m$ we have

$$-f_m(t) = (-1)^m \big(\mathbf{e}^m \wedge \psi(t)\big) \llcorner \mathbf{e}_m(\mathbf{e}_1, \ldots, \mathbf{e}_{m-1}) = (-1)^m \psi(t)(\mathbf{e}_1, \ldots, \mathbf{e}_{m-1});$$

moreover, for any $s \in \mathbf{U}^{m-1}$ we have that

$$(-1)^m {\mathbf{i}_{m-1,m}}^{\#}\psi(s)(\mathbf{e}_1, \ldots, \mathbf{e}_{m-1}) = \psi(\mathbf{i}_{m-1,m}(s))(\mathbf{e}_1, \ldots, \mathbf{e}_{m-1})$$

so (2) holds. $\square$

Stokes' Theorem. Suppose $M \in \mathbf{M}_{m,n}$ and $\mathbf{s}$ is an orientation for M and $\partial\mathbf{s}$ orients ∂M. Then

$$\int_M \omega \;=\; \int_{\partial M} d\omega \quad \text{whenever } \omega \in \mathcal{A}_0^{m-1}(V).$$

Proof. Let $\mathcal{A}$ be an admissible subfamily of $\mathcal{Q}(M, V)$. We have

$$d\omega = d\Big(\sum_{(U,\phi,\chi)\in\mathcal{A}} \chi\Big)\omega = d \sum_{(U,\phi,\chi)\in\mathcal{A}} \chi\omega = \sum_{(U,\phi,\chi)\in\mathcal{A}} d(\chi\omega)$$

so

$$\begin{aligned}
\int_M d\omega &= \sum_{(U,\phi,\chi)\in\mathcal{A}} \int_M d(\chi\omega) \\
&= \sum_{(U,\phi,\chi)\in\mathcal{A}} \mathbf{s_o}(U,\phi) \int_{\phi^{-1}[M]} \phi^{\#} d(\chi\omega)(t)(\mathbf{e_1}, \ldots, \mathbf{e}_m)\, dt \\
&= \sum_{(U,\phi,\chi)\in\mathcal{A}} \mathbf{s_o}(U,\phi) \int_{\phi^{-1}[M]} d\big(\phi^{\#}(\chi\omega)\big)(t)(\mathbf{e_1}, \ldots, \mathbf{e}_m)\, dt.
\end{aligned}$$

We have

$$\omega = \Big(\sum_{(U,\phi,\chi)\in\mathcal{A}} \chi\Big)\omega = \sum_{(U,\phi,\chi)\in\mathcal{A}} \chi\omega;$$

keeping in mind that $(U, \phi \circ \mathbf{i}_{m-1,m}) \in \mathcal{P}(\partial M, V)$ whenever $(U,\phi) \in \mathcal{P}(M,V)$ we find that

$$\begin{aligned}
\int_{\partial M} \omega &= \sum_{(U,\phi,\chi)\in\mathcal{A}} \int_{\partial M} \chi\omega \\
&= \sum_{(U,\phi,\chi)\in\mathcal{A}} \mathbf{s}_{\partial\mathbf{o}}(U, \phi \circ \mathbf{i}_{m-1,m}) \int_{\phi^{-1}[\partial M]} (\phi \circ \mathbf{i}_{m-1,m})^{\#}(\chi\omega)(t)(\mathbf{e_1}, \ldots, \mathbf{e}_{m-1})\, dt \\
&= \sum_{(U,\phi,\chi)\in\mathcal{A}} (-1)^m \mathbf{s_o}(U,\phi) \int_{\phi^{-1}[\partial M]} \mathbf{i}_{m-1,m}{}^{\#}\big(\phi^{\#}(\chi\omega)\big)(t)(\mathbf{e_1}, \ldots, \mathbf{e}_{m-1})\, dt.
\end{aligned}$$

Suppose $(U,\phi,\chi) \in \mathcal{Q}(M,V)$. Then exactly one of the following holds:

$$\begin{aligned}
\phi^{-1}[M] = \mathbf{U}^m \quad &\text{and} \quad \phi^{-1}[\partial M] = \emptyset; \\
\phi^{-1}[M] = \mathbf{U}^{m,m,+} \quad &\text{and} \quad \phi^{-1}[\partial M] = \mathbf{U}^{m-1,m}; \\
\phi^{-1}[M] = \emptyset \quad &\text{and} \quad \phi^{-1}[\partial M] = \emptyset.
\end{aligned}$$

Applying the previous Proposition with ψ there equal $\phi^{\#}(\chi\omega)$ we find that

$$\int_{\phi^{-1}[M]} d\big(\phi^{\#}(\chi\omega)\big)(t)(\mathbf{e_1}, \ldots, \mathbf{e}_m)\, dt = (-1)^m \int_{\phi^{-1}[\partial M]} \mathbf{i}_{m-1,m}{}^{\#}\big(\phi^{\#}(\chi\omega)\big)(t)(\mathbf{e_1}, \ldots, \mathbf{e}_{m-1})\, dt.$$

□