

1. THE REAL NUMBERS.

1.1. Ordered rings.

Definition 1.1. By an **ordered commutative ring with unity** we mean an ordered sextuple

$$(R, +, 0, *, 1, P)$$

such that $(R, +, 0, *, 1)$ is a commutative ring with unity and such that P is a subset of R with the following properties:

(i) if $x \in R$ the exactly one of the following holds:

$$-x \in P; \quad x = 0; \quad x \in P;$$

(ii) if $x \in P$ and $y \in P$ then $x + y \in P$;

(iii) if $x \in P$ and $y \in P$ then $xy \in P$.

If $x \in P$ we say x is **positive** and if $-x \in P$ we say x is **negative**.

We will frequently say “ R is an ordered ring with positive elements P ” instead of “ $(R, \alpha, 0, \mu, 1, P)$ is an ordered ring”.

Suppose R is an ordered ring with positive elements P .

Proposition 1.1. Suppose $x, y \in R$ and $xy = 0$. Then either $x = 0$ or $y = 0$.

Remark 1.1. That is, R is an **integral domain**.

Proof. We have

$$\begin{aligned} x \in P \text{ and } y \in P &\Rightarrow xy \in P \Rightarrow xy \neq 0; \\ x \in P \text{ and } -y \in P &\Rightarrow -(xy) = x(-y) \in P \Rightarrow xy \neq 0; \\ -x \in P \text{ and } y \in P &\Rightarrow -(xy) = (-x)y \in P \Rightarrow xy \neq 0; \\ -x \in P \text{ and } -y \in P &\Rightarrow xy = (-x)(-y) \in P \Rightarrow xy \neq 0. \end{aligned}$$

Thus $x = 0$ or $y = 0$. □

Definition 1.2. We let

$$<$$

be the set of $(x, y) \in R \times R$ such that $y - x \in P$.

Proposition 1.2. $<$ linearly orders R and $P = \{x \in R : x > 0\}$.

Remark 1.2. In particular, if $a \in R$ then exactly one of the following holds:

$$-a > 0; \quad a = 0; \quad a > 0.$$

(Don't forget our convention about $\leq, >$, etc.)

Proof. We leave this as a straightforward exercise for the reader. □

Theorem 1.1. Suppose $a, b, c \in R$. Then

- (i) $a > 0 \Leftrightarrow -a < 0$;
- (ii) $a < 0 \Leftrightarrow -a > 0$;
- (iii) $a \neq 0 \Rightarrow a^2 > 0$;
- (iv) $1 > 0$;
- (v) $a < b \Leftrightarrow a + c < b + c$;

(vi) $a < b$ and $c > 0 \Rightarrow ac < bc$;

Proof. Exercise for the reader. \square

Theorem 1.2. There is one and only one function

$$f : \mathbb{Z} \rightarrow R$$

such that

(1) $\text{rng } f \neq \{0\}$ and

(2) $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ whenever $x, y \in \mathbb{Z}$.

Moreover,

(3) $f(x) < f(y)$ whenever $x, y \in \mathbb{Z}$ and $x < y$.

Remark 1.3. We will frequently not distinguish between the integer n and its image $f(n)$ in R .

Remark 1.4. By virtue of this Theorem, the ring integers modulo a nonzero integer m cannot be ordered.

Proof. By induction, there is one and only function $f^+ : \mathbb{N} \rightarrow R$ such that $f^+(0) = 0$ and $f^+(n + 1) = f^+(n) + 1$ whenever $n \in \mathbb{N}$. Using induction and the fact that the sum of positive members of R is positive we infer that $f^+(n) > 0$ whenever $n \in \mathbb{N}^+$. Using induction again one shows that f^+ preserves arithmetic operations and order.

Now let

$$f = \{(m - n, f^+(m) - f^+(n)) : (m, n) \in \mathbb{N} \times \mathbb{N}\}.$$

We leave it to the reader to verify that f is a function with the desired properties. \square

Definition 1.3. An ordered field is an ordered ring which is a field.

Proposition 1.3. If F is the field of quotients of R then

$$\left\{ \frac{x}{y} : x, y \in P \right\}$$

is a set of positive elements for an ordered field structure on F .

Proof. We leave this as a straightforward exercise for the reader. \square

Now suppose F is an ordered field.

Theorem 1.3. There is one and only one function

$$g : \mathbb{Q} \rightarrow F$$

such that

(1) $\text{rng } g \neq \{0\}$ and

(2) $g(x + y) = g(x) + g(y)$ and $g(xy) = g(x)g(y)$ whenever $x, y \in \mathbb{Z}$.

Moreover,

(3) $g(x) < g(y)$ whenever $x, y \in \mathbb{Q}$ and $x < y$.

Remark 1.5. We will frequently not distinguish between the rational number q and its image $f(q)$ in F .

Proof. Let $f : \mathbb{Z} \rightarrow F$ be such that $\text{rng } f \neq \{0\}$ and $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ whenever $x, y \in \mathbb{Z}$. Let

$$g = \left\{ \left(\frac{x}{y}, \frac{f(x)}{f(y)} \right) : (x, y) \in \mathbb{Z} \times (\mathbb{Z} \sim \{0\}) \right\}.$$

We leave it to the reader to verify that g is a function which has the desired properties. \square

Theorem 1.4. Suppose $a \in F$. Then

$$a > 0 \Leftrightarrow \frac{1}{a} > 0.$$

Proof. Exercise for the reader. \square

Theorem 1.5. Suppose $a, b \in F$ and

$$a \leq b + \epsilon \quad \text{for all } \epsilon > 0.$$

Then

$$a \leq b.$$

Remark 1.6. This need *not* be true over an ordered ring. For example $1 \leq 0 + \epsilon$ for every positive integer ϵ .

Proof. Suppose, contrary to the Theorem, $b < a$. Let $\epsilon = (a - b)/2 > 0$. Then

$$a - (b + \epsilon) = a - \left(b + \frac{a - b}{2} \right) = \frac{a - b}{2} > 0$$

so $b + \epsilon < a$ which is a contradiction. \square

Proposition 1.4. The the following are equivalent:

- (i) for each $R \in F$ with $R > 0$ there is $N \in \mathbb{N}^+$ such that $R < N$;
- (ii) for each $\epsilon \in F$ with $\epsilon > 0$ there is $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \epsilon$.

Proof. Suppose (i) holds and $\epsilon > 0$. There is then $N \in \mathbb{N}^+$ such that $\frac{1}{\epsilon} < N$ so $\frac{1}{N} < \epsilon$ so (ii) holds. In like manner one shows that (ii) implies (i). \square

Definition 1.4. We say an ordered field is **Archimedean** if either of the equivalent conditions in the previous Proposition hold.

Theorem 1.6. Suppose F is an Archimedean ordered field.

- (i) Whenever $c, \epsilon \in F$ and $\epsilon > 0$ there exists a unique integer m such that $m\epsilon \leq c < (m + 1)\epsilon$.

- (ii) Whenever $a, b \in F$ and $a < b$ there is a rational number r such that $a < r < b$.

Proof. To prove (i) we suppose $c \geq 0$ and leave the case $c < 0$ to the reader. The set $A = \{n \in \mathbb{N}^+ : c/\epsilon < n\}$ is nonempty and therefore has a least element l . Let $m = l - 1$. Evidently, $c/\epsilon < l = m + 1$ so $c < (m + 1)\epsilon$. Since l is the least element of A we have $m = l - 1 \notin A$ which amounts to $c/\epsilon \geq m$ so $m\epsilon \leq c$.

To prove (ii), choose a positive integer n such that $1/n < b - a$ and then use (i) to choose an integer m such that $m/n \leq a < (m + 1)/n$. Then

$$b = a + (b - a) > a + 1/n \geq m/n + 1/n = (m + 1)/n$$

so we may take $r = (m + 1)/n$. \square

Example 1.1. The field of rational functions over \mathbb{Q} which is, by definition, the field of quotients of the polynomials over \mathbb{Q} , is an ordered field (Where did the order come from?) which is *not* Archimedean.

There is also the subject of *nonstandard analysis* where one tries to make a field with infinitesimals in it. We'll stick to standard analysis.

1.2. Completely ordered fields.

Definition 1.5. By a **completely ordered field** we mean an ordered field whose ordering is complete.

Theorem 1.7. Any completely ordered field is Archimedean.

Proof. Suppose F is a completely ordered field, $R \in F$ and $R > 0$. If there is no positive integer N such that $R < N$ then R is an upper bound for the positive integers which must therefore have a least upper bound L . Then would then be a positive integer n such that $L - 1 < n$ which implies $L < n + 1$ which is incompatible with L being an upper bound for the set of positive integers. \square

Theorem 1.8. Suppose F is a completely ordered field and $x \in F$. Then

$$x = \sup\{r \in \mathbb{Q} : r < x\}.$$

Proof. Let $X = \{r \in \mathbb{Q} : r < x\}$. By Theorem 1.6(ii) there is $q \in \mathbb{Q}$ such that $x - 1 < q < x$ so X is not empty. Since x is an upper bound for X there is a least upper bound w for X . Since x is an upper bound for X we have $w \leq x$. Were it the case that $w < x$ we could Theorem 1.6(ii) to obtain a rational number q such that $w < q < x$. As $q \in X$ this is incompatible with w being an upper bound for X ; so $w = \sup X = x$. \square

Theorem 1.9. The uniqueness of completely ordered fields. Suppose F_i , $i = 1, 2$, are completely ordered fields. There is one and only function

$$f : F_1 \rightarrow F_2$$

such that

- (i) $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ whenever $x, y \in F_1$ and
- (ii) $f(x) < f(y)$ whenever $x, y \in F_1$ and $x < y$.

Moreover,

$$f(\sup A) = \sup f[A]$$

for any nonempty subset A of X_1 which has an upper bound.

Proof. For $x \in F_1$ let $f(x)$ be the supremum in F_2 of $\{r \in \mathbb{Q} : r < x\}$. We leave it as a straightforward but tedious exercise to verify f has the desired properties. Its uniqueness is evident. \square

Exercise 1.1. Suppose F is a completely ordered field, n is a positive integer, $x \in F$ and $x > 0$. As an exercise you will show that there is $w \in F$ such that $w > 0$ and $w^n = x$. Let

$$W = \{v \in F : v \geq 0 \text{ and } v^n < x\}.$$

Note that $0 \in W$ so $W \neq \emptyset$. Show that W has an upper bound so that $w = \sup W$ exists. Show that $w^n = x$.

Let me get you started. Suppose $w^n \neq x$. If $w^n < x$ come up with $\epsilon > 0$ such that $(w + \epsilon)^n < x$ thereby obtaining a contradiction; in a similar fashion, if $w^n > x$ come up with $\epsilon > 0$ such that $(w - \epsilon)^n > x$. You will need the binomial theorem. You might try the case $n = 2$ first.

So if there is a completely ordered field it is unique structurally. But is there one? In the next two subsections we will give two rather different constructions of a completely ordered field.

Theorem 1.10. Suppose F is an ordered field. Show that the following are equivalent:

- (i) F is complete.
- (ii) If I is an interval then one of the following holds:
 - (a) $I = \emptyset$;
 - (b) $I = \{d\}$ for some $d \in F$;
 - (c) $I = (a, b)$ for some $a, b \in F$ with $a < b$;
 - (d) $I = [a, b)$ for some $a, b \in F$ with $a < b$;
 - (e) $I = (a, b]$ for some $a, b \in F$ with $a < b$;
 - (f) $I = [a, b]$ for some $a, b \in F$ with $a < b$;
 - (g) $I = (-\infty, c)$ for some $c \in F$;
 - (h) $I = (-\infty, c]$ for some $c \in F$;
 - (i) $I = (c, -\infty)$ for some $c \in F$;
 - (j) $I = [c, \infty)$ for some $c \in F$;
 - (k) $I = F$.

Proof. This follows readily from the Exercise in which you were asked to show that F was complete if and only if the only initial segments are

$$\{w \in F : w < x\} \quad \text{and} \quad \{w \in F : w \leq x\}$$

corresponding to $x \in F$. \square

1.3. Dedekind cuts.

Definition 1.6. Dedekind Cuts. We say a subset D of \mathbb{Q} is a **Dedekind cut** if D is an initial segment and D has no largest element.

We let

$$\overline{\mathbb{R}}$$

be the family of Dedekind cuts and we call the members of $\overline{\mathbb{R}}$ **extended real numbers**.

The following simple Proposition will allow us to define a natural linear ordering on $\overline{\mathbb{R}}$.

Proposition 1.5. Suppose $D, E \in \overline{\mathbb{R}}$ and $E \sim D \neq \emptyset$. Then $D \subset E$.

Proof. Suppose $p \in D$. Choose a member q of $E \sim D$. Were it the case that $q < p$ we would have $q \in D$ since D is a Dedekind cut. Thus $p < q$ so $p \in E$ since E is a Dedekind cut. \square

Suppose $D, E \in \overline{\mathbb{R}}$. We declare

$$D < E \quad \text{if } D \neq E \text{ and } D \subset E.$$

It is obvious that $<$ is partial ordering of $\overline{\mathbb{R}}$ and the previous Proposition implies $<$ is trichotomous so $<$ is a linear ordering of $\overline{\mathbb{R}}$. We let

$$-\infty = \emptyset \quad \text{and we let } \infty = \mathbb{Q}.$$

Evidently,

$$-\infty, \infty \in \overline{\mathbb{R}} \quad \text{and} \quad -\infty \leq D \leq \infty \quad \text{whenever } D \in \overline{\mathbb{R}}.$$

In particular, $-\infty$ is a lower bound and ∞ is an upper bound for any subset of $\overline{\mathbb{R}}$.

Proposition 1.6. $<$ is a complete linear ordering of $\overline{\mathbb{R}}$.

Proof. Suppose \mathcal{A} is a nonempty subfamily of $\overline{\mathbb{R}}$. We leave it as a straight forward exercise for the reader to verify that $\cup \mathcal{A}$ is a Dedekind cut. If $A \in \mathcal{A}$ then $A \subset \cup \mathcal{A}$ so $\cup \mathcal{A}$ is an upper bound for \mathcal{A} . If $E \in \overline{\mathbb{R}}$ and E is an upper bound for \mathcal{A} then $A \subset E$ whenever $A \in \mathcal{A}$ so $\cup \mathcal{A} \subset E$. Thus $\cup \mathcal{A}$ is the least upper bound for \mathcal{A} . \square

If $A \subset \overline{\mathbb{R}}$ we let

$$\inf A \quad \text{and} \quad \sup A$$

be the greatest lower bound or *infimum* of A and least upper bound or *supremum* of A , respectively. Note that

$$\inf \emptyset = \infty \quad \text{and} \quad \sup \emptyset = -\infty.$$

It is generally a good idea to exclude the possibility of $\inf \emptyset$ or $\sup \emptyset$ occurring in an argument.

Recall that

$$\mathbb{I}(q) = \{p \in \mathbb{Q} : p < q\}.$$

We leave it as a straight forward exercise for the reader to verify that $\mathbb{Q} \ni r \mapsto \mathbb{I}(r)$ carries \mathbb{Q} univalently into $\overline{\mathbb{R}}$ and that $\mathbb{I}(p) < \mathbb{I}(q)$ if $p, q \in \mathbb{Q}$ and $p < q$.

We let

$$\mathbb{R} = \overline{\mathbb{R}} \sim \{-\infty, \infty\}$$

and we call the members of \mathbb{R} **real numbers**. Note that if $D \in \overline{\mathbb{R}}$ then $D \in \mathbb{R}$ if and only if D is nonempty and has an upper bound in \mathbb{Q} .

We define a field structure on \mathbb{R} as follows. Suppose $D, E \in \mathbb{R}$. We let

$$D + E = \{d + e : d \in D \text{ and } e \in E\}.$$

If $0 \in D \cap E$ we let

$$DE = \{p \in \mathbb{Q} : \text{for some } q, r, q \in D, r \in E, q > 0, r > 0 \text{ and } p < qr\}.$$

We leave it to the reader to modify this definition of multiplication appropriately to treat the case when $0 \notin D \cap E$. It is then a straightforward but rather tedious exercise to verify that the ordered field axioms are satisfied where the additive neutral element is $\mathbb{I}(0)$ and the multiplicative neutral element is $\mathbb{I}(1)$. It is probably a waste of time to do this exercise. See what follows for an alternate construction of the real number systems in which the field properties directly result.

We extend $+$ and $*$ as follows: For any $a \in \mathbb{R}$ we set

$$\begin{aligned} \pm\infty + a &= \pm\infty = a + \pm\infty \\ a(\pm\infty) &= \pm\infty = (\pm\infty)a \quad \text{if } a > 0 \\ a(\pm\infty) &= \mp\infty = (\pm\infty)a \quad \text{if } -a < 0 \end{aligned}$$

and we set

$$0(\pm\infty) = 0 = (\pm\infty)0 \quad \text{if } -\infty < a < 0$$

All other arithmetic operations involving $\pm\infty$ are undefined.

The following summarizes some absolutely fundamental properties of \inf and \sup .

Theorem 1.11. Suppose A and B are nonempty subsets of $\overline{\mathbb{R}}$. Then

- (i) $\inf A \leq \sup A$;
- (ii) if $A \subset B$ then $\inf B \leq \inf A$ and $\sup A \leq \sup B$;
- (iii) $\sup A \leq \inf B$ if and only if $a \leq b$ whenever $a \in A$ and $b \in B$;
- (iv) if $b \in \mathbb{R}$ then $\inf A \leq b$ if and only if for each $\epsilon > 0$ there is $a \in A$ such that $a \leq b + \epsilon$;
- (v) $b \leq \inf A$ if and only if $b \leq a$ for all $a \in A$;
- (vi) $b \leq \sup A$ if and only if for each $\epsilon > 0$ there is $a \in A$ such that $b \leq a + \epsilon$;
- (vii) $\sup A \leq b$ if and only if $a \leq b$ for all $a \in A$.

Proof. Exercise for the reader. □

Theorem 1.12. Suppose $a \in \mathbb{R}$ and B is a nonempty subset of \mathbb{R} . Then

- (i) $\inf(a + B) = a + \inf B$ and $\sup(a + B) = a + \sup B$.
- (ii) $\inf aB = a \inf B$ and $\sup aB = a \sup B$ if $a > 0$.
- (iii) $\inf aB = a \sup B$ and $\sup aB = a \inf B$ if $a < 0$.

Proof. Since $a + b \leq a + \sup B$ for each $b \in B$ we have $\sup(aB) \leq a + \sup B$. Let $\epsilon > 0$. Choose $b \in B$ such that $\sup B \leq b + \epsilon$. Then $a + \sup B \leq a + b + \epsilon \leq \sup(a + B) + \epsilon$. Owing to the arbitrariness of ϵ we infer that $a + \sup B \leq \sup(a + B)$. Thus $\sup(a + B) = a + \sup B$. In a similar fashion one proves that $\inf(a + B) = a + \inf B$. Thus (i) is proved.

Suppose $a > 0$. Since $b \leq \sup B$ for $b \in B$ we have $ab \leq a \sup B$ for $b \in B$ so $\sup(aB) \leq a \sup B$. Let $\epsilon > 0$. Choose $b \in B$ such that $\sup B \leq b(1 + \epsilon/a)$. Then $a \sup B \leq ab(1 + \epsilon/a) = ab + \epsilon$. Owing to the arbitrariness of ϵ we infer that $a \sup B \leq \sup(aB)$. Thus $\sup(aB) = a \sup B$. In a similar fashion one proves that $\inf(ab) = a \inf B$. Thus (ii) is proved.

We leave it to the reader to show that $\inf(-B) = -\sup B$ and that $\sup(-B) = -\inf B$. Using this and (ii) one easily proves (iii) □

An alternate method for defining the real numbers is as follows. In this construction a field structure is an immediate result but the order structure is obtained less directly than with Dedekind cuts.

1.4. Cauchy sequences of rational numbers. We now give another construction of the real numbers.

We say a sequence c of rational numbers is **Cauchy** if for each positive integer n there is a positive integer N such that $|c_i - c_j| < 1/n$ provided $i \geq N$ and $j \geq N$. We say a sequence z of rational numbers is **null** if for each positive integer n there is a positive integer N such that $|z_i| < 1/n$ if $i \geq N$. We set

$$\mathcal{C} = \{c : C \text{ is a Cauchy sequence of rational numbers}\}$$

and we set

$$\mathcal{N} = \{z : z \text{ is a null sequence of rational numbers}\}.$$

It is then a simple matter to verify that \mathcal{C} has the structure of an integral domain with respect to pointwise addition and multiplication of rational numbers and that \mathcal{N} is a maximal ideal in \mathcal{C} . The quotient space \mathcal{C}/\mathcal{N} is therefore a field. One orders this quotient space by declaring an element A to be less than B if there is positive rational number r such that whenever s and t are representatives of A and B , respectively, there is a positive integer N such that

$$s_i + r < t_i \text{ whenever } i \geq N.$$

It is then a straightforward matter to verify that this completely orders the field \mathcal{C}/\mathcal{N} .

2. LIMITS OF SEQUENCES IN $\overline{\mathbb{R}}$.

Definition 2.1. Suppose a is a sequence in $\overline{\mathbb{R}}$ and $L \in \overline{\mathbb{R}}$. We say a_n **approaches** L as n **tends to infinity** and write

$$\lim_{n \rightarrow \infty} a_n = L$$

if one of the following holds.

(i) $-\infty < L < \infty$ and whenever $0 < \epsilon < \infty$ there is $N \in \mathbb{R}$ such that

$$n \geq N \Rightarrow |a_n - L| \leq \epsilon;$$

(ii) $L = -\infty$ and whenever $-\infty < R < 0$ there is $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow a_n \leq R;$$

(iii) $L = \infty$ and whenever $0 < R < \infty$ there is $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow a_n \geq R.$$

Theorem 2.1. Suppose a is a sequence in $\overline{\mathbb{R}}$. Then

$$\lim_{n \rightarrow \infty} a_n = \sup \text{rng } a$$

if a is nondecreasing and

$$\lim_{n \rightarrow \infty} a_n = \inf \text{rng } a$$

if a is nonincreasing.

Proof. Suppose a is a nonincreasing and $L = \sup \mathbf{rng} a$.

Suppose $-\infty < L < \infty$. Let $\epsilon > 0$. Choose a nonnegative integer N such that $L \leq a_N + \epsilon$. If $n \geq N$ we have $L - \epsilon \leq a_N \leq a_n \leq L$ which implies $|a_n - L| \leq \epsilon$. Thus $\lim_{n \rightarrow \infty} a_n = L$.

Suppose $L = \infty$. Let $R \in \mathbb{R}$. Choose a nonnegative integer N such that $a_N \geq R$. If $n \geq N$ we have $a_n \geq a_N \geq R$. Thus $\lim_{n \rightarrow \infty} a_n = L$.

Suppose $L = -\infty$. Let $R \in \mathbb{R}$. Choose a nonnegative integer N such that $a_N \leq R$. If $n \geq N$ we have $a_n \leq a_N \leq R$. Thus $\lim_{n \rightarrow \infty} a_n = L$.

One treats the case when a is nonincreasing in a similar fashion; we leave the details to the reader. \square

As an example of the above, we have the following.

Theorem 2.2. Suppose $0 < r < 1$. Then $\lim_{n \rightarrow \infty} r^n = 0$.

Proof. Proof One. Suppose $\epsilon > 0$. Since $1 - r > 0$ we may invoke the Archimedean Property of the real numbers to choose a positive integer M such that $1/(M + 1) < 1 - r$. Using the Archimedean property once again we obtain a positive integer N such that $N \geq M/\epsilon$. We will show that

$$n \in \mathbb{N} \text{ and } n \geq N \Rightarrow r^n < \epsilon.$$

Note that $r < M/(M + 1)$. For any positive integer n we use the binomial theorem to infer that

$$\left(\frac{M+1}{M}\right)^n \geq \frac{M^n + nM^{n-1}}{M^n} = 1 + \frac{n}{M}$$

so

$$r^n < \left(\frac{M}{M+1}\right)^n \leq \frac{1}{1 + \frac{n}{M}} < \frac{M}{n}.$$

It follows that

$$r^n < \frac{M}{n} \leq \frac{M}{N} \leq \epsilon \quad \text{if } n \geq N.$$

\square

Proof. Proof Two. **This proof will not use the Archimedean property.**

Let $a_n = r^n$ for $n \in \mathbb{N}$. Then a is a decreasing sequence in $(0, 1)$ so

$$L = \inf \mathbf{rng} a = \lim_{n \rightarrow \infty} a_n$$

exists and $L \in [0, 1)$. Since

$$\mathbf{rng} a = \{r^n : n \in \mathbb{N}\} = \{1\} \cup \{r^{n+1} : n \in \mathbb{N}^+\} = \{1\} \cup \mathbf{rng} ra$$

we find that $\inf \mathbf{rng} ra = \inf \mathbf{rng} a$. Making use of previous work we find that

$$rL = r \inf \mathbf{rng} a = \inf(\mathbf{rng} ra) = \inf \mathbf{rng} ra = \inf \mathbf{rng} a = L.$$

Were it the case that $L \neq 0$ we would have $r = 1$ so $L = 0$. \square

Definition 2.2. Suppose a is a sequence in $\overline{\mathbb{R}}$. We let

$$\liminf_{n \rightarrow \infty} a_n = \sup_n \inf_{m \geq n} a_m$$

and we let

$$\limsup_{n \rightarrow \infty} a_n = \inf_n \sup_{m \geq n} a_m.$$

It is very important to bear in mind that

$$\mathbb{N} \ni n \mapsto \inf\{a_m : m \geq n\} \quad \text{is nondecreasing}$$

and that

$$\mathbb{N} \ni n \mapsto \sup\{a_m : m \geq n\} \quad \text{is nonincreasing.}$$

2.1. Relating \lim , \liminf **and** \limsup . Throughout this subsection we fix a sequence a in \mathbb{R} and let

$$\underline{l} = \liminf_{n \rightarrow \infty} a_n \quad \text{and} \quad \bar{l} = \limsup_{n \rightarrow \infty} a_n.$$

Theorem 2.3. $\underline{l} \leq \bar{l}$.

Proof. For each $n \in \mathbb{N}$ we let $A_n = \{a_m : m \geq n\}$. We have

$$(1) \quad \inf A_n \leq \sup A_n, \quad n \in \mathbb{N}.$$

I claim that

$$(2) \quad \inf A_{n_1} \leq \sup A_{n_2}, \quad n_1, n_2 \in \mathbb{N}.$$

If $n_1 = n_2$ then (2) follows from (1). If $n_1 < n_2$ then $\inf A_{n_1} \leq \inf A_{n_2}$ because $A_{n_2} \subset A_{n_1}$ and $\inf A_{n_2} \leq \sup A_{n_1}$ by (1) so (2) holds. If $n_1 > n_2$ then $\inf A_{n_1} \leq \sup A_{n_1}$ by (1) and $\sup A_{n_1} \leq \sup A_{n_2}$ because $A_{n_1} \subset A_{n_2}$ so (2) holds.

Let $I = \{\inf A_n : n \in \mathbb{N}\}$ and let $S = \{\sup A_n : n \in \mathbb{N}\}$. Since each member of I is less than or equal each member of S we infer that $\underline{l} = \sup I \leq \inf S = \bar{l}$. \square

Proposition 2.1. Suppose $-\infty < l < \infty$. Then following statements hold:

- (i) $l \leq \underline{l}$ if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $l < a_n + \epsilon$ whenever $n \geq N$.
- (ii) $\underline{l} \leq l$ if and only if for each $\epsilon > 0$ and each $N \in \mathbb{N}$ there is $n \geq N$ such that $a_n \leq l + \epsilon$.
- (iii) $l < \bar{l}$ if and only if for each $\epsilon > 0$ and each $N \in \mathbb{N}$ there is $n \geq N$ such that $l \leq a_n + \epsilon$.
- (iv) $\bar{l} \leq l$ if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $a_n < l + \epsilon$ whenever $n \geq N$.

Proof. Exercise for the reader. \square

Proposition 2.2. The following statements hold.

- (i) $\underline{l} = -\infty$ if and only if for all $R \in \mathbb{R}$ and all $N \in \mathbb{N}$ there exist $n \in \mathbb{N}$ such that $n \geq N$ and $a_n \leq R$.
- (ii) $\underline{l} = \infty$ if and only if for all $R \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $R \leq a_n$ if $n \geq N$.
- (iii) $\bar{l} = \infty$ if and only if for all $R \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n \geq R$ for some $n \geq N$.
- (iv) $\bar{l} = -\infty$ if and only if for all $R \in \mathbb{R}$ there is $N \in \mathbb{N}$ $a_n \leq R$ if $n \geq N$.

Proof. Exercise for the reader. \square

Theorem 2.4. Suppose $L \in \overline{\mathbb{R}}$. Then

$$\underline{l} = L = \bar{l} \Leftrightarrow \lim_{n \rightarrow \infty} a_n = L.$$

Proof. Exercise for the reader. \square

Definition 2.3. We say a sequence a in \mathbb{R} is **Cauchy** if for each $\epsilon > 0$ there is a nonnegative integer N such that

$$m, n \in \mathbb{N} \text{ and } m, n \geq N \Rightarrow |a_m - a_n| \leq \epsilon.$$

Theorem 2.5. Suppose a is a sequence in \mathbb{R} . The following are equivalent:

- (i) $\lim_{n \rightarrow \infty} a_n = L$ for some $L \in \mathbb{R}$.
- (ii) a is Cauchy.

Remark 2.1. We will give a proof that is independent of the material on \liminf and \limsup . Note that L in the proof is none other than $\liminf_{n \rightarrow \infty} a_n$.

Proof. Suppose (i) holds. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon/2$ if $n \geq N$. Then if $m, n \geq N$ we have

$$|a_m - a_n| \leq |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus a is Cauchy.

Now suppose (ii) holds. For each $n \in \mathbb{N}$ let $T_n = \{a_m : m \in \mathbb{N} \text{ and } m \geq n\}$ and let $I_n = \inf T_n$. Since $T_m \subset T_n$ if $m, n \in \mathbb{N}$ and $m \geq n$ we find that I is nondecreasing. Let $L = \sup\{I_n : n \in \mathbb{N}\}$.

Suppose $\epsilon > 0$ and suppose $0 < \eta < \epsilon$. Choose $N \in \mathbb{N}$ such that $m, n \in \mathbb{N}$,

$$m \geq N \text{ and } n \geq N \Rightarrow |a_m - a_n| < \eta.$$

Suppose $n \in \mathbb{N}$ and $n \geq N$. Then

$$a_m \geq a_n - \eta \quad \text{if } m \in \mathbb{N} \text{ and } m \geq N.$$

This implies

$$L \geq I_n \geq a_n - \eta.$$

We also have

$$I_m \leq a_m \leq a_n + \eta \quad \text{if } m \in \mathbb{N} \text{ and } m \geq N.$$

Since I is nondecreasing, this implies

$$L \leq a_n + \eta.$$

We infer that $-\infty < L < \infty$ and that $|L - a_n| \leq \eta < \epsilon$, as desired. \square

Exercise 2.1. Suppose a and b are sequences in $\overline{\mathbb{R}}$. Show that

$$\liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n \leq \liminf_{n \rightarrow \infty} a_n + b_n$$

and that

$$\limsup_{n \rightarrow \infty} a_n + b_n \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

Exhibit sequences a and b in $\overline{\mathbb{R}}$ such that equality does not hold in either case.

Exercise 2.2. Suppose

$$a_1, a_2, \dots, a_n, \dots, \quad n = 1, 2, 3, \dots$$

is a bounded sequence in \mathbb{R} with limit 0 and let s be the sequence of **arithmetic means**:

$$s_n = \frac{1}{n}(a_1 + \dots + a_n), \quad n = 1, 2, 3, \dots$$

Show that s has limit 0.

Exercise 2.3. Let

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}, \quad n = 1, 2, 3, \dots$$

Show that a converges. Bonus: To what?

Exercise 2.4. Suppose a is a sequence in $\overline{\mathbb{R}}$ with limit $L \neq 0$. Show that

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{L}.$$

3. THE UNCOUNTABILITY OF THE REAL NUMBERS.

Definition 3.1. If a is a sequence in $[0, \infty]$ we let

$$\sum_{n=0}^{\infty} a_n = \sup \left\{ \sum_{n=0}^N a_n : N \in \mathbb{N} \right\} = \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n.$$

Theorem 3.1. Suppose $r \in \mathbb{R} \sim \{1\}$. Then

$$(1) \quad \sum_{m=0}^n r^m = \frac{1 - r^{n+1}}{1 - r} \quad \text{for } n \in \mathbb{N}.$$

Proof. The (1) holds trivially if $n = 0$. Let us suppose, inductively, that the $n \in \mathbb{N}$ and (1) holds. Then

$$\sum_{m=0}^{n+1} r^m = \sum_{m=0}^n r^m + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1} = \frac{1 - r^{(n+1)+1}}{1 - r}.$$

□

Corollary 3.1. If s is a sequence in $[0, 1]$ then

$$(2) \quad \sum_{n=0}^{\infty} \frac{s_n}{2^{n+1}} \leq 1.$$

Definition 3.2. A basic map. We let

\mathcal{S}

be the set of $s \in \{0, 1\}^{\mathbb{N}}$ such that $\{n \in \mathbb{N} : s(n) = 1\}$ is infinite and, keeping in mind (2), we define

$$\Phi : \mathcal{S} \rightarrow (0, 1]$$

by letting

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{s_n}{2^{n+1}} \quad \text{for } s \in \mathcal{S}.$$

We endow \mathcal{S} with the **dictionary order** by declaring $s < t$ for $s, t \in \mathcal{S}$ if $s \neq t$ and $s_n = 0$ and $t_n = 1$ where n is the least member of $\{p \in \mathbb{N} : s_p \neq t_p\}$.

Proposition 3.1. $<$ is a linear ordering of \mathcal{S} .

Proof. It follows directly from (1) that $<$ is trichotomous.

Suppose $s, t, u \in \mathcal{S}$, $s < t$ and $t < u$. Let n be the least member of $\{p \in \mathcal{N} : s_p \neq t_p\}$ and let o be the least member of $\{p \in \mathcal{N} : t_p \neq u_p\}$. Then $s_n = 0$ and $t_n = 1$ and $t_o = 0$ and $u_o = 1$. In particular, it is impossible that $n = o$. In case $n < o$ we have $s_p = t_p = u_p$ for $p < n$, $s_n = 0$ and $t_n = 1$, and $t_n = u_n$ so $s < u$. In case $o < n$ we have $s_p = t_p = u_p$ for $p < o$, $s_o = t_o$ and $t_o = 0$ and $u_o = 1$ so $s < u$. Thus $<$ is transitive. \square

Proposition 3.2. Suppose $s, t \in \mathcal{S}$. Then $s < t$ if and only if $\Phi(s) < \Phi(t)$.

Proof. Let o be the least member of $\{p \in \mathbb{N} : s_p \neq t_p\}$ and let

$$B = \sum_{n < o} \frac{s_n}{2^{n+1}}.$$

Then

$$\Phi(s) = B + \frac{s_o}{2^{o+1}} + S \quad \text{and} \quad \Phi(t) = B + \frac{t_o}{2^{o+1}} + T$$

where we have set

$$S = \sup \left\{ \sum_{n=o+1}^N \frac{s_n}{2^{n+1}} : N \in \mathbb{N} \text{ and } o+1 \leq N \right\}$$

and

$$T = \sup \left\{ \sum_{n=o+1}^N \frac{t_n}{2^{n+1}} : N \in \mathbb{N} \text{ and } o+1 \leq N \right\}.$$

Then $S \leq 1/2^{o+1}$ by Theorem 3.1 and $T > 0$ since $\{n \in \mathbb{N} : n > o+1 \text{ and } t_n \neq 0\}$ is nonempty.

Suppose $s < t$. Then $s_o = 0$ and $t_o = 1$ so

$$\Phi(s) = B + S < B + \frac{1}{2^{o+1}} + T = \Phi(t).$$

Suppose $\Phi(s) < \Phi(t)$. Then $s \neq t$. By the trichotomy of $<$ we have either $s < t$ or $t < s$. Were it the case that $t < s$ we would have $\Phi(t) < \Phi(s)$ by what we proved in the previous paragraph. So $s < t$ \square

Proposition 3.3. $\text{rng } \Phi = (0, 1]$.

Proof. We have already shown that $\text{rng } \Phi \subset (0, 1]$.

Suppose $a \in (0, 1]$. We define the sequence $p : \mathbb{N} \rightarrow \mathbb{N}$ by induction as follows. p_0 is the least member of \mathbb{N} such that $2^{-m-1} < a$. If $n \in \mathbb{N}$ and $p_j, j \in \mathbb{I}(n)$, are defined we let p_n be the least member m of \mathbb{N} such that

$$\sum_{j \in \mathbb{I}(n)} 2^{-(p_j+1)} + 2^{-(m+1)} < a.$$

We let $s \in \mathcal{S}$ be such that $\{n \in \mathbb{N} : s_n = 1\} = \text{rng } p$. We leave it to the reader to use the Archimedean property of the real numbers to verify that $\Phi(s) = a$. \square

We are now able to prove the following fundamental result.

Theorem 3.2. $2^{\mathbb{N}} \approx \mathbb{R}$.

Proof. Since

$$2^{\mathbb{N}} \ni A \mapsto 1_A \in \{0, 1\}^{\mathbb{N}}$$

is a univalent function with domain $2^{\mathbb{N}}$ and range $\{0, 1\}^{\mathbb{N}}$ it will suffice to show that $\{0, 1\}^{\mathbb{N}} \approx \mathbb{R}$.

For each $n \in \mathbb{N}$ let $\lambda_n : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ be such that

$$\lambda_n(s)(j) = s(n+j) \quad \text{for } j \in \mathbb{N}$$

note that $\lambda_n[\mathcal{S}] = \mathcal{S}$.

For each $n \in \mathbb{N}$ let \mathcal{S}_n be the set of $s \in \mathcal{S} \sim \{1_{\mathbb{N}}\}$ such that

$$s_n = 0 \quad \text{and} \quad s_j = 0 \text{ if } j \in \mathbb{I}(n).$$

Obviously, $\{1_{\mathbb{N}}\} \cup \{\mathcal{S}_n : n \in \mathbb{N}\}$ is a partition of \mathcal{S} . For each $n \in \mathbb{N}$ let $\lambda_n : \mathcal{S}_n \rightarrow \{0, 1\}^{\mathbb{N}}$ be such that

$$\lambda_n(s)(j) = s(n+1+j) \quad \text{for } j \in \mathbb{N}.$$

Clearly, λ_n is univalent with range equal \mathcal{S} and this implies that $\Phi \circ \lambda_n$ is univalent with range $(0, 1]$.

For each $n \in \mathbb{N}$ let $\mathbf{t}_n : [0, 1) \rightarrow \mathbb{R}$ be such that

$$\mathbf{t}_n(x) = \begin{cases} j+x & \text{if } n = 2j \text{ for some } j \in \mathbb{N}, \\ -j-x & \text{if } n = 2j+1 \text{ for some } j \in \mathbb{N} \end{cases} \quad \text{for } x \in [0, 1).$$

Thus $\mathbf{t}_n \circ \Phi \circ \lambda_n$ has domain \mathcal{S}_n and carries \mathcal{S}_n univalently onto

$$\begin{cases} (j, j+1] & \text{if } n = 2j \text{ for some } j \in \mathbb{N}, \\ [-(j+1), -j) & \text{if } n = 2j+1 \text{ for some } j \in \mathbb{N}. \end{cases}$$

It follows that

$$\{(1_{\mathbb{N}}, 0)\} \cup \bigcup \{\mathbf{t}_n \circ \Phi \circ \lambda_n : n \in \mathbb{N}\}$$

is a univalent function with domain $\{0, 1\}^{\mathbb{N}}$ and range \mathbb{R} . □

Remark 3.1. It is not too hard to use the fact that $\mathcal{S} \approx (0, 1]$ together with the Schroeder-Bernstein Theorem to show that $2^{\mathbb{N}} \approx \mathbb{R}$.

4. ABSOLUTE VALUES.

For $a \in R$ we set

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{else} \end{cases}$$

and note that

$$|a| + a \geq 0 \quad \text{and} \quad |a| - a \geq 0.$$

Note that

$$(1) \quad |a| \leq b \Leftrightarrow a \leq b \quad \text{and} \quad -a \leq b$$

whenever $a, b \in R$.

Lemma 5.1. Suppose ν is a positive integer and $a \in [0, \infty]^{\mathbb{J}(0, \nu)}$. Then

$$(3) \quad \bigvee_{\xi=0}^{\nu} a_{\xi} + \sum_{\mu=0}^{\nu-1} a_{\mu} \wedge \left(\bigvee_{\xi=\mu+1}^{\nu} a_{\xi} \right) = \sum_{\mu=0}^{\nu} a_{\mu}, \quad a \in [0, \infty]^{\nu}.$$

Proof. (??) follows from ?? if $\nu = 1$. The proof may be completed by a simple induction on ν the details of which we leave to the reader. \square