1. The integers and the rational numbers.

1.1. The integers. Let
\[ Z = \{ ((m, n), (p, q)) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) : m + q = p + n \}. \]
It is a simple matter to verify that \( Z \) is an equivalence relation on \( \mathbb{N} \times \mathbb{N} \). Let
\[ Z = (\mathbb{N} \times \mathbb{N})/Z \]
and call its members integers. Let
\[ m - n = (m, n)/Z \quad \text{for} \quad (m, n) \in \mathbb{N} \times \mathbb{N}. \]

One easily verifies that
\[ \mathbb{N} \ni n \mapsto (n - 0)/Z \]
is univalent. In what follows we will not distinguish between a member of \( \mathbb{N} \) and its image under this mapping.

One verifies that there is a unique unary operation \(-\) on \( Z \) such that
\[ -(m - n) = n - m \]
and that there are unique binary operation + and * on \( Z \) such that
\[ (m-n) + (p-q) = (m+p) - (n+q) \quad \text{and} \quad (m-n) * (p-q) = (m*p + n*q) - (m*q + n*p) \]
for \( (m, n), (p, q) \in \mathbb{N} \times \mathbb{N} \); on the right hand side of these equations + and * are the operations on \( \mathbb{N} \).

One easily verifies that \((Z, +, 0, -)\) is an Abelian group and that \((Z, +, 0, *)\) is an integral domain in which 1 is the neutral element with respect to *.

1.2. The rational numbers. Let
\[ Q = \{ ((m, n), (p, q)) \in (\mathbb{Z} \times (\mathbb{Z} \sim \{0\})) \times (\mathbb{Z} \times (\mathbb{Z} \sim \{0\})) : m*q = n*p \}. \]
It is a simple matter to verify that \( Q \) is an equivalence relation on \( \mathbb{Z} \times (\mathbb{Z} \sim \{0\}) \). Let
\[ Q = (\mathbb{Z} \times (\mathbb{Z} \sim \{0\}))/Q \]
and call its members rational numbers. Let
\[ \frac{m}{n} = (m, n)/Q \quad \text{for} \quad (m, n) \in \mathbb{Z} \times (\mathbb{Z} \sim \{0\}). \]

One easily verifies that
\[ \mathbb{Z} \ni n \mapsto \left( \frac{n}{1} \right)/Q \]
is univalent. In what follows we will not distinguish between a member of \( \mathbb{Z} \) and its image under this mapping.

One verifies that there is a unique unary operation \(-\) on \( \mathbb{Z} \) such that
\[ \frac{-m}{n} = \frac{-m}{n} \quad \text{for} \quad (m, n) \in \mathbb{Z} \times (\mathbb{Z} \sim \{0\}) \]
and that there are unique binary operation + and * on \( \mathbb{Z} \) such that
\[ \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq} \quad \text{and} \quad \frac{m}{n} * \frac{p}{q} = \frac{m*p}{n*q} \]
for \( (m, n), (p, q) \in \mathbb{N} \times \mathbb{N} \); on the right hand side of these equations -, + and * are the operations on \( \mathbb{Z} \).

One easily verifies that and that \((Q, +, 0, *, 1)\) is an field.

Note that if one replaces \( \mathbb{Z} \) by any integral domain \( D \) this construction results in a field which is called the field of quotients of \( D \).