Homotopies and the Poincaré Lemma.

Let $I = [0,1]$ and let T be the vector field on $\mathbf{R} \times \mathbf{R}^n$ which assigns the vector $(1,0)$ to each point of $\mathbf{R} \times \mathbf{R}^n$. Let p and q be the projections of $\mathbf{R} \times \mathbf{R}^n$ on $\mathbf{R}$ and $\mathbf{R}^n$, respectively. For each $t \in \mathbf{R}$ let $i_t : \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$ assign (t,x) to $x \in \mathbf{R}^n$.

Proposition. Suppose ω is a smooth m-form defined on some open subset of $\mathbf{R} \times \mathbf{R}^n$. Then

$$d(\iota_T \omega) + \iota_T(d\omega) = \partial_T \omega.$$

Proof. We have

$$d\omega \;=\; p \wedge \partial_T \omega + \sum_{j=1}^n \mathbf{e}^j \circ q \wedge \partial_{(0,\mathbf{e}_j)} \omega. \tag{1}$$

Moreover, if $U \in \mathbf{R} \times \mathbf{R}^n$ then

$$\iota_T \partial_U \omega \;=\; \partial_U \iota_T \omega. \tag{2}$$

It follows from (1), (2) and the way interior multiplication interacts with wedge that

$$\iota_T(d\omega) \;=\; \partial_T \omega - \partial_T(\iota_T \omega) - \sum_{j=1}^n \mathbf{e}^j \circ q \wedge \partial_{(0,\mathbf{e}_j)}(\iota_T \omega) \;=\; \partial_T \omega - d(\iota_T \omega).$$

□

Definition. Suppose ω is a smooth m-form defined on some open subset V of $\mathbf{R} \times \mathbf{R}^n$. Let

$$V_I = \{x \in \mathbf{R}^n : (t,x) \in V \text{ whenever } t \in I\}.$$

Let

$$\omega_I \in \mathcal{A}^{m-1}(V_I)$$

be such that

$$\omega_I(x) = \int_0^1 i_t^\#(\iota_T \omega)(x)\, dt$$

for $x \in V_I$.

Lemma. Suppose ω and V are as in the previous Definition. Then

$$d(\omega_I) + (d\omega)_I = i_1^\# \omega - i_0^\# \omega.$$

Proof. For any $(t, \mathbf{x}) \in \mathbf{R} \times \mathbf{R}^n$ we have

$$i_t^\#(d(\iota_T \omega) + \iota_T(d\omega))(\mathbf{x}) \;=\; d(i_t^\# \iota_T \omega)(\mathbf{x}) + i_t^\# \iota_T(d\omega)(\mathbf{x})$$

and

$$i_t^\# \partial_T \omega(\mathbf{x}) \;=\; \frac{d}{dt}(i_t^\# \omega)(\mathbf{x}).$$

Integrate from $t = 0$ to $t = 1$ and make use of the preceding Proposition. □

The Poincaré Lemma. Suppose U is an open subset of $\mathbf{R}^n$ wihich is **contractible**; this means, by definition, that there are a map

$$h : I \times U \to U$$

and a point $\mathbf{a}$ in U such that h has a smooth extension to an open set containing $I \times U$,

(1) $h(1, \mathbf{p}) = \mathbf{p}$ whenever $\mathbf{p} \in U$ and

(2) $h(0, \mathbf{p}) = \mathbf{a}$ whenever $\mathbf{p} \in U$.

Then any smooth **closed** m-form on U is **exact**. That is, if $\omega \in \mathcal{A}^m(U)$ and $d\omega = 0$ then there is $\eta \in \mathcal{A}^{m-1}(U)$ such that

$$\omega = d\eta.$$

Proof. By the result of (3) we have

$$d((h^{\#}\omega)_I) + (d(h^{\#}\omega))_I = i_1^{\#}h^{\#}\omega - i_0^{\#}h^{\#}\omega.$$

But $d(h^{\#}\omega)) = h^{\#}(d\omega) = 0$, $h \circ i_1$ is the identity map of U so $i_1^{\#}h^{\#}\omega = \omega$ and $h \circ i_0$ is constant so $i_0^{\#}h^{\#}\omega = 0$. Thus we may set $\eta = (h^{\#}\omega)_I$. $\square$