Homotopies and the Poincaré Lemma.

Let $I = [0, 1]$ and let $T$ be the vector field on $\mathbb{R} \times \mathbb{R}^n$ which assigns the vector $(1, 0)$ to each point of $\mathbb{R} \times \mathbb{R}^n$. Let $p$ and $q$ be the projections of $\mathbb{R} \times \mathbb{R}^n$ on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. For each $t \in \mathbb{R}$ let $i_t : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ assign $(t, x)$ to $x \in \mathbb{R}^n$.

**Proposition.** Suppose $\omega$ is a smooth $m$-form defined on some open subset of $\mathbb{R} \times \mathbb{R}^n$. Then
\[ d(i_T \omega) + i_T (d\omega) = \partial_T \omega. \]

**Proof.** We have

\[ d\omega = p \wedge \partial_T \omega + \sum_{j=1}^n e_j \circ q \wedge \partial_{(0,e_j)} \omega. \]

Moreover, if $U \in \mathbb{R} \times \mathbb{R}^n$ then

\[ i_T \partial_U \omega = \partial_U i_T \omega. \]

It follows from (1), (2) and the way interior multiplication interacts with wedge that
\[ i_T (d\omega) = \partial_T \omega - \partial_T (i_T \omega) - \sum_{j=1}^n e_j \circ q \wedge \partial_{(0,e_j)} (i_T \omega) = \partial_T \omega - d(i_T \omega). \]

\[ \square \]

**Definition.** Suppose $\omega$ is a smooth $m$-form defined on some open subset $V$ of $\mathbb{R} \times \mathbb{R}^n$. Let

\[ V_I = \{ x \in \mathbb{R}^n : (t, x) \in V \text{ whenever } t \in I \}. \]

Let

\[ \omega_I \in \mathcal{A}^{m-1}(V_I) \]

be such that
\[ \omega_I(x) = \int_0^1 i_t^# (i_T \omega)(x) \, dt \]

for $x \in V_I$.

**Lemma.** Suppose $\omega$ and $V$ are as in the previous Definition. Then
\[ d(\omega_I) + (d\omega)_I = i_1^# \omega - i_0^# \omega. \]

**Proof.** For any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ we have
\[ i_t^# (d(i_T \omega) + i_T (d\omega))(x) = d(i_t^# i_T \omega)(x) + i_t^# i_T (d\omega)(x) \]

and
\[ i_t^# \partial_T \omega(x) = \frac{d}{dt}(i_t^# \omega)(x). \]

Integrate from $t = 0$ to $t = 1$ and make use of the preceding Proposition. \[ \square \]

**The Poincaré Lemma.** Suppose $U$ is an open subset of $\mathbb{R}^n$ which is contractible; this means, by definition, that there is a map
\[ h : I \times U \to U \]
and a point \( a \) in \( U \) such that \( h \) has a smooth extension to an open set containing \( I \times U \),

1. \( h(1, p) = p \) whenever \( p \in U \) and
2. \( h(0, p) = a \) whenever \( p \in U \).

Then any smooth \textbf{closed} \( m \)-form on \( U \) is \textbf{exact}. That is, if \( \omega \in \mathcal{A}^m(U) \) and \( d\omega = 0 \) then there is \( \eta \in \mathcal{A}^{m-1}(U) \) such that

\[
\omega = d\eta.
\]

\textbf{Proof.} By the result of (3) we have

\[
d((h^\# \omega)_I) + (d(h^\# \omega))_I = i_1^\# h^\# \omega - i_0^\# h^\# \omega.
\]

But \( d(h^\# \omega) = h^\# (d\omega) = 0 \), \( h \circ i_1 \) is the identity map of \( U \) so \( i_1^\# h^\# \omega = \omega \) and \( h \circ i_0 \) is constant so \( i_0^\# h^\# \omega = 0 \). Thus we may set \( \eta = (h^\# \omega)_I \). \( \square \)