Partitions of Unity.

Theorem. Suppose $a \in \mathbf{R}^n$ and $0 < r < s < \infty$. Then there is a smooth function

$$\psi : \mathbf{R}^n \to [0,1]$$

such that

$$\mathbf{B}_a(r) \subset \mathbf{int}\{x \in \mathbf{R}^n : \psi(x) = 1\} \quad \text{and} \quad \mathbf{spt}\,\psi \subset \mathbf{U}_a(s).$$

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be such that

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have already shown that f is smooth. Let $g : \mathbf{R} \to \mathbf{R}$ be such that $g(x) = f(x-2)f(3-x)$ for $x \in \mathbf{R}$; note that g is smooth, that g vanishes outside $(2,3)$ and that g is positive on $(2,3)$. Let $h : \mathbf{R} \to \mathbf{R}$ be such that

$$h(x) = \frac{\int_x^\infty g(t)\,dt}{\int_{-\infty}^\infty g(t)\,dt}$$

for $x \in \mathbf{R}$; note that h equals 1 on $(-\infty, 2]$, that h is between 0 and 1 on $(2,3)$ and that h equals 0 on $[3,\infty)$. Let $\zeta : \mathbf{R} \to \mathbf{R}$ be the affine function (this amounts to saying that the graph is a straight line) such that $\zeta(r) = 1$ and $\zeta(s) = 4$. Finally, let

$$\psi(x) = h(\zeta(|x-a|)) \quad \text{for } x \in \mathbf{R}^n.$$

□

Theorem. Suppose K is a compact subset of $\mathbf{R}^n$, U is an open subset of $\mathbf{R}^n$ and $K \subset U$. Then there is a smooth compactly supported function

$$\chi : \mathbf{R}^n \to [0,1]$$

such that

$$K \subset \mathbf{int}\{x \in \mathbf{R}^n : \chi(x) = 1\} \quad \text{and} \quad \mathbf{spt}\,\chi \subset U.$$

Proof. Let

$$\mathcal{U} = \{\mathbf{U}_a(r) : a \in K,\ 0 < r, \infty \ \text{ and } \mathbf{B}_a(2r) \subset U\}.$$

Since K is compact and $\mathcal{U}$ is an open covering of K there are a positive integer m, points $a_1, \ldots, a_m$ in K and radii $r_1, \ldots, r_m$ such that

$$K \subset \bigcup_{i=1}^m \mathbf{U}_{a_i}(r_i) \quad \text{and} \quad \bigcup_{i=1}^m \mathbf{B}_{a_i}(2r_i) \subset U$$

for each $i = 1, \ldots, m$. Make use of the previous Theorem to choose for each $i = 1, \ldots, m$ a smooth function $\chi_i : \mathbf{R}^n \to [0,1]$ such that $\mathbf{B}_{a_i} \subset \mathbf{int}\,\{x \in \mathbf{R}^n : \chi(x) = 1\}$ and $\mathbf{spt}\,\chi_i \subset \mathbf{B}_{a_i}(2r_i)$. Finally, we let

$$\chi = 1 - \prod_{i=1}^m (1 - \chi_i).$$

□

Definition. Suppose $\mathcal{U}$ is a family of open subsets of $\mathbf{R}^n$. By a **partition of unity subordinate to** $\mathcal{U}$ we mean a function which assigns to each member α of some set A a smooth compactly supported function

$$\chi_\alpha : \mathbf{R}^n \to [0,1]$$

such that

(1) for each $\alpha \in A$ there is $U \in \mathcal{U}$ such that $\mathbf{spt}\, \chi_\alpha \subset U$;

(2) for each compact subset K of $\bigcup \mathcal{U}$ the set

$$\{\alpha \in A : \mathbf{spt}\, \chi_\alpha \cap K \neq \emptyset\}$$

is finite;

(3) $\sum_{\alpha \in A} \chi_\alpha(x) = 1$ for each $x \in \bigcup \mathcal{U}$.

Theorem. The existence of partitions of Unity. Suppose $\mathcal{U}$ is a family of open subsets of $\mathbf{R}^n$. Then there is a partition of unity subordinate to $\mathcal{U}$.

Proof. For each nonegative integer m let $\mathcal{C}_m$ be the family of cubes we introduced in the first semester. For each $m = 0, 1, 2, \ldots$ we define the subfamily $\mathcal{D}_m$ of $\mathcal{C}_m$ as follows. We let

$$\mathcal{D}_0 = \{C \in \mathcal{C}_0 : \mathbf{cl}(C) \subset U \text{ for some } U \in \mathcal{U}\}$$

and we let

$$\mathcal{D}_{m+1} = \{C \in \mathcal{C}_{m+1} \sim \bigcup_{i=0}^{m} \mathcal{D}_i : \mathbf{cl}(C) \subset U \text{ for some } U \in \mathcal{U}\}.$$

We set

$$\mathcal{D} = \bigcup_{m=0}^{\infty} \mathcal{D}_m.$$

Note that

(4) $\mathcal{D}$ is disjointed with union $\bigcup \mathcal{U}$ and

(5) any compact subset of U meets only finitely many members of $\mathcal{D}$.

For each $D \in \mathcal{D}$ let

$$\hat{D} = \bigcup \{C \in \mathcal{D} : \mathbf{cl}(C) \cap \mathbf{cl}(D) \neq \emptyset\}.$$

Note that for any $D \in \mathcal{D}$

(6) $\mathbf{cl}(D) \subset \mathbf{int}(\hat{D})$;

(7) $\hat{D}$ is the union of a finite subfamily of $\mathcal{D}$.

For each $D \in \mathcal{D}$ use Theorem 2 to choose a smooth function $\psi_D : \mathbf{R}^n \to [0, 1]$ such that, for some $U \in \mathcal{U}$,

$$D \subset \{x \in \mathbf{R}^n : \psi_D(x) = 1\} \quad \text{and} \quad \mathbf{spt}(\psi_D) \subset U \cap \hat{D}.$$

For each $D \in \mathcal{D}$ let

$$\chi_D = \frac{\psi_D}{\sum_{C \in \mathcal{D}} \psi_C}.$$

Let $\mathcal{A}$ above equal $\mathcal{D}$. $\square$