Orientation.

Let n be a positive integer, let m be a positive integer not exceeding n and let V be an n-dimensional vector space.

Associated subspaces. For each $\xi \in \bigwedge_m V$ we let

$$\mathbf{Ass}(\xi) = \{v \in V : v \wedge \xi = 0\}$$

and note that $\mathbf{Ass}(\xi)$ is a linear subspace of V which we call the **subspace associated to** ξ**.**

Proposition. Suppose $\xi \in \bigwedge_m V \sim \{0\}$. Then $\mathbf{dim}\,\mathbf{Ass}(\xi) \le m$. Moreover, if $l = \mathbf{dim}\,\mathbf{Ass}(\xi) > 0$ and $v_1, \ldots, v_l$ is a basis for $\mathbf{Ass}(\xi)$ then $l \le m$ and

$$\xi = v_1 \wedge \ldots v_l \wedge \eta$$

for some $\eta \in \bigwedge_{m-l} V$.

Proof. Suppose $l = \mathbf{dim}\,\mathbf{Ass}(\xi) > 0$ and $v_1, \ldots, v_n$ be a basis for V such that $v_1, \ldots, v_l$ is a basis for $\mathbf{Ass}(\xi)$. Write

$$\xi = \sum_{\lambda \in \mathbf{alt}(m,n)} v^\lambda(\xi) v_\lambda.$$

For each $i = 1, \ldots, l$ we have

$$0 = v_i \wedge \xi = \sum_{\lambda \in \mathbf{alt}(m,n),\ i \notin \mathbf{rng}\,\lambda} v^\lambda(\xi) v_i \wedge v_\lambda$$

which implies that $v^\lambda(\xi) = 0$ if $i \notin \mathbf{rng}\,\lambda$. $\Box$

Definition. Suppose m is a positive integer and $\xi \in \bigwedge_m V$. We say ξ is **decomposable** or **simple** if there are $v_1, \ldots, v_m \in V$ such that $\xi = v_1 \wedge \cdots \wedge v_m$. In view of the preceding Proposition, ξ is decomposable if and only if $\mathbf{dim}\,\mathbf{Ass}(\xi) = m$.

Example. Let $\xi = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4 \in \bigwedge_2 \mathbf{R}^4$. If $x \in \mathbf{R}^4$ we have

$$x \wedge \xi = x_3\, \mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 + x_4\, \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge + x_1\, \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 + x_2\, \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4.$$

It follows that $\mathbf{Ass}(\xi) = \{0\}$.

Suppose $M \in \mathbf{M}_m(V)$.

Definition. We let

$$\mathcal{O}(M)$$

be the set of continuous maps

$$\mathbf{o} : M \to \bigwedge_m V$$

such that

$$\mathbf{Ass}(\mathbf{o}(a)) = \mathbf{Tan}_a(M) \quad \text{for each } a \in M.$$

We say $\mathbf{o}$ is an **orienting** m**-vector field for** M if $\mathbf{o} \in \mathcal{O}(M)$. We say M is **orientable** if $\mathcal{O}(M) \ne \emptyset$.

Definition. Suppose M is orientable. Whenever $\mathbf{o}_i \in \mathcal{O}(M)$, $i = 1, 2$, we let

$$\mathbf{c}(\mathbf{o}_1, \mathbf{o}_2) : M \to \mathbf{R} \sim \{0\}$$

be such that

$$\mathbf{o}_1(a) = \mathbf{c}(\mathbf{o}_1, \mathbf{o}_2)(a)\mathbf{o}_2(a) \quad \text{whenever } a \in M$$

and note that $\mathbf{c}(\mathbf{o}_1, \mathbf{o}_2)$ is continuous. Let

$$\mathbf{o}(M) = \{(\mathbf{o}_1, \mathbf{o}_2) \in \mathcal{O}(M) \times \mathcal{O}(M) : \mathbf{c}(\mathbf{o}_1, \mathbf{o}_2) > 0\}$$

and note that $\mathbf{o}(M)$ is an equivalence relation on $\mathcal{O}(M)$; an **orientation of** M is, by definition, an equivalence class with respect to $\mathbf{o}(M)$ If $\mathbf{o} \in \mathbf{O}(M)$ we call the equivalence class of $\mathbf{o}$ with respect to $\mathbf{o}(M)$ the **orientation of** M **induced by** $\mathbf{o}$.

Definition. Suppose $M \in \mathbf{M}_n(\mathbf{R}^n)$; that is, M is an open subset of $\mathbf{R}^n$. Then

$$M \ni a \mapsto \mathbf{e}_1 \wedge \cdots \mathbf{e}_n \in \{\xi \in \bigwedge_m \mathbf{R}^n : |\xi| = 1\}$$

is an orienting vector field for M and we call the induced orientation the **standard orientation of** M.

Unit normals to hypersurfaces. Suppose $m = n - 1$ and

$$\mathbf{N} : M \to \mathbf{S}^{n-1}$$

is a continuous map such that

$$\{t\mathbf{N}(a) : t \in \mathbf{R}\} = \mathbf{Nor}_a(M).$$

Let

$$\mathbf{o}(a) = (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) \llcorner \beta\big(\mathbf{N}(a)\big) \quad \text{for } a \in M.$$

Then $\mathbf{o}$ is an orienting vector field for M and

$$\mathbf{N}(a) \wedge \mathbf{o}(a) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \quad \text{for } a \in M.$$

We call orientation of M induced by $\mathbf{o}$ the **standard orientation of** M.

Orienting a boundary. Suppose $1 \le m$ and $M \in \mathbf{M}_m(V)$.

Theorem. There is one and only one map

$$\mathbf{n} : \partial M \to \mathbf{S}^{n-1}$$

such that

$$-\mathbf{n}(b) \in \mathbf{Nor}_b(\partial M) \cap \mathbf{Tan}_b(M).$$

Proof. This is a straightforward consequence of the definitions. □

Definition. The map $\mathbf{n}$ in the preceding Theorem is called the **outward pointing unit normal to** M **along** ∂M.

Theorem. Suppose $\mathbf{o}$ is an orientation for M. Then there is one and only one orienting vector field $\partial\mathbf{o}$ of ∂M such that

$$\lim_{M \ni a \to b} \mathbf{o}(a) = \mathbf{n}(b) \wedge \partial\mathbf{o}(b) \quad \text{whenever } b \in \partial M.$$

Proof. Exercise for the reader. The point here is that if $(\mathbf{U}^n, \Phi, U) \in \mathbf{Diffeo}_n$ is such that $U \subset V$, $\Phi(0) = b$ and $U \cap M = \Phi[\mathbf{U}^{n,m,+}]$ then there is $s \in \{-1, 1\}$ such that

$$|\bigwedge_m \partial\Phi(t)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m)| = s\mathbf{o}(\Phi(t))\big(\bigwedge_m \partial\Phi(t)\big)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m) \quad \text{for } t \in U.$$

Moreover,

$$\partial\Phi(0)(\mathbf{e}_m) \bullet \mathbf{n}(b) < 0.$$

□

Definition. We call the orientation of ∂M induced by $\partial\mathbf{o}$ the **orientation of** ∂M **induced by** $\mathbf{o}$.

The torus and the Möbius band. Let J be the skewsymmetry of $\mathbf{R}^3$ such that

$$J(\mathbf{e}_1) = \mathbf{e}_2, \quad J(\mathbf{e}_2) = -\mathbf{e}_1, \quad J(\mathbf{e}_3) = 0.$$

Note that

$$e^{\theta J}, \; \theta \in \mathbf{R}$$

rotates $\mathbf{R}^3$ by θ radians in the right-handed sense around the x_3-axis. For each $\phi \in \mathbf{R}$ let

$$U(\phi) = \cos\phi\, \mathbf{e}_1 + \sin\phi\, \mathbf{e}_3 \quad \text{and} \quad V(\phi) = -\sin\phi\, \mathbf{e}_1 + \cos\phi\, \mathbf{e}_3, \; \phi \in \mathbf{R};$$

note that $U' = V$, that $\{U(\theta), \mathbf{e}_2, V(\theta)\}$ is an orthonormal basis for $\mathbf{R}^3$ and that

$$U(\theta) \wedge \mathbf{e}_2 \wedge V(\theta)\} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3, \quad \theta \in \mathbf{R}.$$

Suppose $0 < R < \infty$. Let $S = \{(\rho, \theta, \phi) \in \mathbf{R}^3 : -R < \rho < R\}$ and let

$$F : S \to \mathbf{R}^3$$

be such that

$$F(\rho, \theta, \phi) = e^{\theta J}\big(R\mathbf{e}_1 + \rho U(\phi)\big), \quad (\rho, \theta, \phi) \in S.$$

Note that F is univalent on the sets

$$(-R, R) \times (a, a + 2\pi) \times (b, b + 2\pi)$$

corresponding to $a, b \in \mathbf{R}$. For any $(\rho, \theta, \phi) \in S$ we have

$$\begin{aligned}
\partial_1 F(\rho, \theta, \phi) &= e^{\theta J}\big(U(\phi)\big),\\
\partial_2 F(\rho, \theta, \phi) &= e^{\theta J}\Big(J\big(R\mathbf{e}_1 + \rho U(\phi)\big)\Big) = e^{\theta J}\Big(\big(R + \rho\cos\phi\big)\mathbf{e}_2\Big),\\
\partial_3 F(\rho, \theta, \phi) &= e^{\theta J}\Big(\rho V(\phi)\Big)
\end{aligned}$$

so

$$\bigwedge\nolimits_3 \partial F(\rho, \theta, \phi)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = \rho(R + \rho\cos\phi) \bigwedge\nolimits_3 e^{\theta J}\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3.$$

Suppose $0 < r < R$.
Set

$$T_r = \{F(r, \theta, \phi) : (\theta, \phi) \in \mathbf{R}^2\}.$$

One calls T_r a **torus**. Evidently, $T_r \in \mathbf{M}_2(\mathbf{R}^3)$. We may define

$$\mathbf{N} : T_r \to \mathbf{S}^2$$

by requiring that

$$\mathbf{N}(F(r, \theta, \phi)) = \partial_1 F(\rho, \theta, \phi) = e^{\theta J}\Big(U(\theta)\Big), \quad (\theta, \phi) \in \mathbf{R}^2.$$

It follows from the foregoing that $\mathbf{N}$ is a unit normal field along T_r; in particular, T_r is orientable.
Set

$$f_r(\rho, \phi) = F(\rho, 2\phi, \phi), \quad (\rho, \phi) \in (-r, r) \times \mathbf{R}$$

and let

$$M_r = \mathbf{rng}\, f_r.$$

One calls M_r a **Möbius band**. Evidently, $M_r \in \mathbf{M}_2(\mathbf{R}^3)$.
Note that

$$f_r(0, 0) = R\mathbf{e}_1 = f_r(0, \pi).$$

We have

$$\partial_1 f(\rho,\phi) = \partial_1 F(\rho, 2\phi, \phi) = e^{2\phi J}\big(U(\phi)\big),$$
$$\partial_2 f(\rho,\phi) = 2\partial_2 F(\rho, 2\phi, \phi) + \partial_3 F(\rho, 2\phi, \phi) = e^{2\phi J}\Big(2\big(R + \rho\cos\phi\big)\mathbf{e}_2 + \rho V(\phi)\Big)$$

for $(\rho,\phi) \in (-r, r) \times \mathbf{R}$. Let

$$\xi(\rho,\phi) = \bigwedge\nolimits_2 \partial f_r(\rho,\phi)(\mathbf{e}_1 \wedge \mathbf{e}_2), \quad (\rho,\phi) \in (-r, r) \times \mathbf{R}.$$

We have

$$\xi(\rho,\phi) = 2\partial_2 F(\rho, 2\phi, \phi) \wedge \partial_3 F(\rho, 2\phi, \phi) = \bigwedge\nolimits_2 e^{2\phi J}\Big(\big(R + \rho\cos\phi\big)\mathbf{e}_2 \wedge \rho V(\phi)\Big)$$

for $(\rho,\phi) \in (-r, r) \times \mathbf{R}$. In particular,

$$\xi(\rho,\phi) \neq 0 \quad \text{for } (\rho,\phi) \in (-r, r) \times \mathbf{R}.$$

Since

$$\xi(0,0) = 2R\mathbf{e}_1 \wedge \mathbf{e}_2 = -\xi(0,\pi)$$

we find that M_r is *not* orientable.