0.1. Definition. Suppose $V_1, \ldots, V_m$ and $W$ are vector spaces. We say a function

$$\mu : V_1 \times \cdots \times V_m \to W$$

is multilinear if it is linear in each of its $m$ arguments when the other $m - 1$ are held fixed. Let

$$\mathcal{L}(V_1, \ldots, V_m; W)$$

be the set of such $\mu$. Note that $\mathcal{L}(V_1, \ldots, V_m; W)$ is a linear subspace of the vector space of all $W$-valued functions on $V_1 \times \cdots \times V_m$ and is thus a vector space with respect to pointwise addition and scalar multiplication.

Suppose $\omega_i \in V_i^*$, $i = 1, \ldots, m$ and $w \in W$. Define

$$\omega_1 \cdots \omega_m : V_1 \times \cdots \times V_m \to W$$

to have the value $\omega_1(v_1) \cdots \omega_m(v_m) w$ at $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ and note that $\omega_1 \cdots \omega_m w \in \mathcal{L}(V_1, \ldots, V_m; W)$.

In case $W = \mathbb{R}$ and $w = 1$ one customarily writes

$$\omega_1 \cdots \omega_m$$

for $\omega_1 \cdots \omega_m w$.

0.2. Problem 1. Suppose for each $i = 1, 2, V_i$ is a finite dimensional vector space of dimension $n_i$ and with ordered basis $v_i$. Let $\mu \in \mathcal{L}(V_1, V_2; \mathbb{R})$ and let $A \in M_{n_2}^{n_1}$ be such that

$$A(i, j) = \mu(v_i, v_j), \quad i = 1, \ldots, n_1, \quad j = 1, \ldots, n_2.$$

Show that there are $\omega_i \in V_i^*$, $i = 1, 2$, such that $\mu = \omega_1 \omega_2$ if and only if the rank of $A$ does not exceed 1.

0.3. Problem 2. Suppose $V_1, \ldots, V_m$ and $W$ are finite dimensional. Let $B_i$ be a basis for $V_i$, $i = 1, \ldots, m$ and let $C$ be a basis for $W$. Show that

$$\mu = \sum_{(v_1, \ldots, v_m, w) \in B_1 \times \cdots \times B_m \times C} w^* (\mu(v_1, \ldots, v_m)) v^*_1 \cdots v^*_m w$$

for each $\mu \in \mathcal{L}(V_1, \ldots, V_m; W)$. Use this to show that

$$\{v^*_1 \cdots v^*_m w : (v_1, \ldots, v_m, w) \in B_1 \times \cdots \times B_m \times C\}$$

is a basis for $\mathcal{L}(V_1, \ldots, V_m; W)$, concluding thereby that its dimension is $n_1 \cdots n_m$.\l

0.4. Definition. Suppose now that $V_i$ has norm $|\cdot|_V$, $i = 1, \ldots, m$ and that $W$ has norm $|\cdot|_W$. For each $\mu \in \mathcal{L}(V_1, \ldots, V_m; W)$ we let

$$||\mu||_{V_1, \ldots, V_m; W} = \sup \{|\mu(v_1, \ldots, v_m)|_W : v_i \in V_i \text{ and } |v_i|_V \leq 1\}.$$ 

Very often one omits the subscripts on the norms relying on the context to resolve the resulting ambiguities.

0.5. Problem 3.

(1) Suppose $\mu \in \mathcal{L}(V_1, \ldots, V_m; W)$ and $M \in [0, \infty)$. Then $|\mu(v_1, \ldots, v_m)| \leq M |v_1| \cdots |v_m|$ whenever $v_i \in V_i$, $i = 1, \ldots, m$ if and only if $||\mu|| \leq M$.

(2) $||\mu + \nu|| \leq ||\mu|| + ||\nu||$ whenever $\mu, \nu \in \mathcal{L}(V_1, \ldots, V_m; W)$;

(3) $||c\mu|| = |c||\mu||$ whenever $c \in \mathbb{R}$ and $\mu \in \mathcal{L}(V_1, \ldots, V_m; W)$;

(4) If $\mu \in \mathcal{L}(V_1, \ldots, V_m; W)$ then $\mu$ is continuous if and only if $||\mu|| < \infty$.\l
0.6. Problem 4. Suppose $U, V, W$ are normed vector spaces, $L \in \mathcal{L}(U; V)$ and $M \in \mathcal{L}(V; W)$. Then

$$||M \circ L|| \leq ||M|| ||L||.$$

0.7. Problem 5. Suppose $V$ is a finite dimensional Euclidean space. Show that the mapping

$$V \ni v \mapsto (V \ni \tilde{v} \mapsto \tilde{v} \cdot v \in \mathbb{R}) \in V^*$$

carries $V$ isomorphically onto $V^*$. This map is called the polarity of the inner product and we induce an inner product on $V^*$ by requiring that it be an isometry. Conversely, if $\beta$ carries $V$ isomorphically onto $V^*$ and satisfies the conditions

(i) $\beta(v)(w) = \beta(w)(v)$, $v, w \in V$ and

(ii) $\beta(v)(v) > 0$ if $v \in V \sim \{0\}$

then we may obtain an inner product on $V$ by setting $v \cdot w = \beta(v)(w), /v, w \in V$.

0.8. Problem 6. Verify that the adjoint mapping defined earlier is a linear isomorphism if $V$ and $W$ above are finite dimensional. Do this by showing that the adjoint mapping is linear (this is trivial) and that if $B$ is a basis for $V$ and $C$ is a basis for $W$ then $\{v^*w : v \in B$ and $w \in C\}$ is a basis for $\mathcal{L}(V, W)$; $\{wv^* : v \in B$ and $w \in C\}$ is a basis for $\mathcal{L}(W^*, V^*)$; and

$$(v^*w)^* = wv^*$$

whenever $v \in B$ and $w \in C$.

(Here we have written $w$ instead of $\iota(w)$ for $w \in W$ as we indicated we might do so when $\iota$ was defined.)

0.9. Definition. Let $V$ and $W$ be finite dimensional Euclidean spaces with polarities $\beta$ and $\gamma$, respectively. Let

$$^* = \beta^{-1} \circ (\cdot^*) \circ \gamma$$

where the $^*$ on the right is the adjoint introduced previously and where the one on the left is being introduced now. Note that $^*$ (on the left!), also called the adjoint (sorry about that, you were warned!) carries $\mathcal{L}(V; W)$ isomorphically onto $\mathcal{L}(W; V)$. Verify that if $L \in \mathcal{L}(V; W)$ and $K \in \mathcal{L}(W; V)$ then

$$L(v) \cdot w = v \cdot K(w)$$

whenever $v \in V, w \in W \iff K = L^*.$

Verify that, under appropriate hypotheses,

$$(L \circ M)^* = M^* \circ L^*.$$
Verify that $\zeta$ is linear. Verify that it is an isomorphism by showing that if $B$ is a basis for $V$ then $\zeta$ carries the basis $\hat{\varepsilon}^* v$ of $L(V; V)$ to the basis $\iota(\hat{\varepsilon}^* v)$ of $L(V^*, V; R)$.

**0.12.** Definition. Suppose $V_1, \ldots, V_m$ are finite dimensional vector spaces. We set

$$V_1 \otimes \cdots \otimes V_m = L(V_1^*, \ldots, V_m^*; R)$$

and call this vector space the tensor product of $V_1, \ldots, V_m$. For each $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ we set

$$v_1 \otimes \cdots \otimes v_m = \iota(v_1) \cdots \iota(v_m) \in V_1 \otimes \cdots \otimes V_m$$

and note that

$$V_1 \times \cdots \times V_m \ni (v_1, \ldots, v_m) \mapsto v_1 \otimes \cdots \otimes v_m \in V_1 \otimes \cdots \otimes V_m$$

is multilinear.

**0.13.** Problem 9. Show that if $V_1, \ldots, V_m$ are finite dimensional vector spaces and $W$ is a vector space then

$$L(V_1 \otimes \cdots \otimes V_m; W) \ni \mu \mapsto (V_1 \times \cdots \times V_m \ni (v_1, \ldots, v_m) \mapsto \mu(v_1 \otimes \cdots \otimes v_m)) \in L(V_1, \ldots, V_m; W)$$

carries $L(V_1 \otimes \cdots \otimes V_m; W)$ isomorphically onto $L(V_1, \ldots, V_m; W)$. In particular, for any $\tilde{\mu} \in L(V_1, \ldots, V_m; W)$ there is one and only $\mu \in L(V_1 \otimes \cdots \otimes V_m; W)$ such that

$$\tilde{\mu}(v_1, \ldots, v_m) = \mu(v_1 \otimes \cdots \otimes v_m), \quad v_i \in V_i, \ i = 1, \ldots, m.$$ 

This is called the universal property of the tensor product.

**0.14.** Problem 10. Suppose $V$ and $W$ are finite dimensional vector spaces. By the universal property of the tensor product there is a unique linear map

$$\gamma : V^* \otimes W \to L(V; W)$$

such that $\gamma(\omega \otimes v) = \omega v$ whenever $\omega \in V^*$ and $w \in W$. Show that $\gamma$ is an isomorphism by finding a basis of $V^* \otimes W$ which is carried to a basis of $L(V; W)$ by $\gamma$.

**0.15.** Problem 11. Suppose $V$ and $W$ are finite dimensional Euclidean spaces. Verify that

$$L(V; W) \times L(V; W) \ni (K, L) \mapsto \text{trace}(K^* \circ L)$$

is an inner product.

Verify that

$$|L| \leq \sqrt{\dim V} ||L|| \quad \text{and that} \quad ||L|| \leq |L|.$$ 

Note that equality occurs in the left hand inequality if $L^* = L^{-1}$ which is to say if $L$ is orthogonal. Note that equality occurs in the right hand inequality if $\dim V = 1$.

**0.16.**

**0.17.** Problem 12. Suppose $V$ and $W$ are finite dimensional Euclidean spaces. Suppose $L \in L(V; W)$. Show that

$$||L^*|| = ||L||.$$ 

Do this by first showing that

$$||L|| = \sup\{|L(v) \cdot w| : v \in V, \ |v| \leq 1, \ w \in W, \ |w| \leq 1\}.$$