1. Metric spaces.

**Definition 1.1.** Let $X$ be a set. We say $\rho$ is a metric on $X$ if

$$\rho : X \times X \to \{ r \in \mathbb{R} : r \geq 0 \}$$

and

(i) $\rho(x, y) = \rho(y, x)$ whenever $x, y \in X$;
(ii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ whenever $x, y, z \in X$.
(iii) If $x, y \in X$ then $\rho(x, y) = 0$ if and only if $x = y$.

The inequality in (ii) is called the triangle inequality. A metric space is an ordered pair $(X, \rho)$ such that $X$ is a set and $\rho$ is a metric on $X$.

We now fix a set $X$ and a metric $\rho$ on $X$.

For each $a \in X$ and each positive real number $r$ we let

$$U^a(r) = \{ x \in X : \rho(x, a) < r \}$$
and we let $B^a(r) = \{ x \in X : \rho(x, a) \leq r \}$.

We say a subset $U$ of $X$ is open if for each $a \in U$ there is a positive real number $\epsilon$ such that $U^a(\epsilon) \subset U$. We leave as an exercise to the reader the proof of the fact that the family of open sets is a topology on $X$. This topology is called the topology induced by the metric $\rho$; one proves this in exactly the same way we proved the corresponding fact for $\mathbb{R}^n$.

Suppose $x$ is a sequence in $X$ and $b \in X$. Note that

$$\lim_{\nu \to \infty} x_\nu = b$$

if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\rho(x_\nu, b) < \epsilon$$
whenever $\nu \in \mathbb{N}$ and $n \geq N$.

**Theorem 1.1.** Suppose $a \in X$ and $r$ is a positive real number. Then $U^a(r)$ is open and $B^a(r)$ is closed.

**Proof.** Suppose $b \in U^a(r)$. I claim that $U^b(r - \rho(a, b)) \subset U^a(r)$. Indeed, suppose $x \in U^b(r - \rho(a, b))$; then, by the triangle inequality and the fact that $\rho(a, b) = \rho(b, a)$,

$$\rho(x, a) \leq \rho(x, b) + \rho(b, a) = \rho(x, a) + \rho(a, b) < (r - \rho(a, b)) + \rho(a, b) = r$$
so $x \in U^a(r)$. Thus $U^a(r)$ is open. The reader should verify that, in a similar fashion, one may prove that $X \sim B^a(r)$ is open so that $B^a(r)$ is closed. □

**Theorem 1.2.** Then the topology induced by the metric $\rho$ is Hausdorff.

**Proof.** Suppose $x, y \in X$ and $x \neq y$. Let $r = \rho(x, y)/2$, note that $r > 0$ and let $U = U^x(r)$ and $V = U^y(r)$. Then, by the previous theorem, $U$ and $V$ are open. Suppose $z \in U$. Then, by the triangle inequality and the fact that $\rho(z, z) = \rho(z, x)$, we infer that

$$\rho(z, y) \geq \rho(x, y) - \rho(x, z) = \rho(x, y) - \rho(z, x) > r - r/2 = r/2$$
so $z \notin V$. Thus $U \cap V = \emptyset$ and this proves that $X$ is Hausdorff. □

**Definition 1.2.** Whenever $A \subset X$ and $x \in X$ we let

$$\rho(a, A) = \inf\{ \rho(x, y) : y \in A \}$$
and we call this number the distance from $a$ to $A$. 

1
Theorem 1.3. Suppose $A$ is a subset of $X$. Then

(i) $|\rho(x,A) - \rho(y,A)| \leq \rho(x,y)$ whenever $x, y \in X$;
(ii) $\text{cl} A = \{x \in X : \rho(x,A) = 0\}$;
(iii) $\text{int} A = \{x \in X : \rho(x, X \sim A) > 0\}$.

Proof. We leave this as an exercise to the reader. In proving (i) one makes use of the fact that if $a, b$ and $c$ are real numbers then $|a - b| \leq c \leftrightarrow a \leq b + c$ and $b \leq a + c$

which implies that $||a| - |b|| \leq |a - b|$.

Definition 1.3. Suppose $A$ is a subset of $X$. We let $\text{diam} A = \sup\{\rho(x,y) : x, y \in A\}$

and call this number the diameter of $A$. We say $A$ is bounded if $\text{diam} A < \infty$.

2. Completeness.

Definition 2.1. We say $(X, \rho)$ is complete (or when it is clear from the context what $\rho$ is that $X$ is complete) if

$$\bigcap C \neq \emptyset$$

whenever $C$ is a nonempty nested family of nonempty closed subsets of $X$ such that

$$\inf\{\text{diam} C : C \in C\} = 0.$$

Note that if $C$ is as in the preceding definition then $\bigcap C$ has exactly one point.

A sequence $x$ in $X$ is a Cauchy sequence if

$$\inf\{\text{diam}\{x_n : m \in \mathbb{N} \text{ and } m \geq n\} : n \in \mathbb{N}\} = 0.$$

This is equivalent to the statement that for each positive real number $\epsilon$ there is a nonnegative integer $N$ such that

$$\rho(x_l, x_m) \leq \epsilon$$

whenever $l \geq N$ and $m \geq N$.

Proposition 2.1. $X$ is complete if and only if every Cauchy sequence converges.

Proof. Suppose $X$ is complete and $x$ is a Cauchy sequence in $X$. For each positive integer $m$ let $C_m = \text{cl}\{x_n : m \in \mathbb{N} \text{ and } m \geq n\}$ and note that $C = \{C_m : m \in \mathbb{N}\}$ is a nonempty nested family of closed subsets of $X$ with the property that

$$\inf\{\text{diam} C : C \in C\} = 0.$$

Because $X$ is complete there is a unique member $l$ of $\bigcap C$. We now show that $l$ is the limit of the sequence $x$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\text{diam} C_N \leq \epsilon$.

If $n \geq N$ then both $l$ and $x_n$ are members of $C_n$ which is a subset of $C_N$ so

$$\rho(x_n, l) \leq \text{diam} C_n \leq \text{diam} C_N \leq \epsilon.$$

On the other hand, suppose $X$ is a metric space in which every Cauchy sequence converges and let $C$ be a nonempty nested family of nonempty closed sets with the property that

$$\inf\{\text{diam} C : C \in C\} = 0.$$

In case there is $C \in C$ such that $\text{diam} C = 0$ then there is $c \in X$ such that $C = \{c\}$ so $\cap C = \{c\}$. So suppose $\text{diam} C > 0$ for each $C \in C$. Choose a
sequence $C$ in $\mathcal{C}$ such that $\text{diam} C_{\nu+1} < \text{diam} C_\nu$ whenever $\nu \in \mathbb{N}$ and such that $\lim_{\nu \to \infty} \text{diam} C_\nu = 0$. Note that $C_{\nu+1} \subset C_\nu$ whenever $\nu \in \mathbb{N}$ for if this were not the case for some $\nu \in \mathbb{N}$ we would have $C_\nu \subset C_{\nu+1}$ since $\mathcal{C}$ is nested and this would imply $\text{diam} C_\nu \leq \text{diam} C_{\nu+1}$. Let $x$ be a sequence in $X$ such that $x_\nu \in C_\nu$ for each $\nu \in \mathbb{N}$. Note that we have used the Axiom of Choice twice. Evidently, $x$ is a Cauchy sequence. Let $c$ be its limit and suppose $B \in \mathcal{C}$. Choose $\nu \in \mathbb{N}$ such that $\text{diam} C_\nu < \text{diam} B$. For any $\mu \in \mathbb{N}$ with $\mu \geq \nu$ we have $C_\mu \subset C_\nu \subset B$ so, by a preceding Theorem
\[
\text{dist} (c, B) \leq \text{dist} (c, C_\mu) \leq \rho(c, x_\mu) + \text{dist} (x_\mu, C_\mu) = \rho(c, x_\mu) \to 0 \quad \text{as} \quad \nu \to \infty.
\]
Thus $\text{dist} (c, B) = 0$ so, again by a preceding Theorem, $c \in \overline{\text{cl} B} = B$ Thus $c \in \cap \mathcal{C}$.

**Proposition 2.2.** Suppose $A \subset X$ and $\sigma = \rho|(A \times A)$. Then $(A, \sigma)$ is complete if and only if $A$ is a closed subset of $X$.

**Proof.** Suppose $(A, \sigma)$ is complete. Let $b$ be a point of the $\rho$-closure of $A$. For each $\varepsilon > 0$ let $C_\varepsilon = A \cap \{x \in X : \rho(x, b) \leq \varepsilon\}$ and note that $C_\varepsilon$ is $\sigma$-closed. Moreover, $\emptyset \neq A \cap \{x \in X : \rho(x, b) < \varepsilon\} \subset A \cap C_\varepsilon$ and $\text{diam} C_\varepsilon \leq 2\varepsilon$ whenever $0 < \varepsilon < \infty$. Thus $C = \{C_\varepsilon : 0 < \varepsilon < \infty\}$ is a nonempty family of nonempty $\sigma$-closed sets; thus there is $c \in A$ such that $\{c\} = \cap C$. It is evident that $b = c$ so $b \in A$ and, therefore, $A$ is $\rho$-closed.

Suppose $A$ is $\rho$-closed. Let $x$ be a Cauchy sequence in $A$. Evidently, $x$ is a Cauchy sequence in $X$. As $(X, \rho)$ is complete there is $b \in X$ such that $\lim_{\nu \to \infty} x_\nu = b$. Since $A$ is $\rho$-closed we infer that $b \in A$. Thus $(A, \sigma)$ is complete. 

**Theorem 2.1.** $\mathbb{R}^n$ is complete.

**Proof.** We have already proved this in the case $n = 1$.

Suppose $x$ is a Cauchy sequence in $\mathbb{R}^n$. For each $i \in \{1, \ldots, n\}$ let $p_i : \mathbb{R}^n \to \mathbb{R}$ assign to $a \in \mathbb{R}^n$ its $i$th coordinate; note that $|p_i(a)| \leq |a|$ whenever $a \in \mathbb{R}^n$. This implies $p_i \circ x$ is a Cauchy sequence in $\mathbb{R}$ for each $i \in \{1, \ldots, n\}$ which, therefore, converges to some $L_i \in \mathbb{R}$. Let $L \in \mathbb{R}^n$ be such that $p_i(L) = L_i$ for $i \in \{1, \ldots, n\}$. Then
\[
|x_\nu - L| \leq \sqrt{n} \max\{|p_i(x) - p_i(L)| : i \in \{1, \ldots, n\}\} \to 0 \quad \text{as} \quad \nu \to \infty.
\]
That is, $\lim_{\nu \to \infty} x_\nu = L$. 

3. THE LEBESGUE RADIUS OF AN OPEN COVERING.

**Definition 3.1.** Suppose $\mathcal{U}$ is a family of open subsets of $X$. $X$. For each $x \in \cup \mathcal{U}$ we let
\[
u \mapsto (r : 0 < r < \infty \text{ and } U^x(r) \subset U \text{ for some } U \in \mathcal{U});
\]
evidently $\nu (x)$ is a nonempty open interval with infimum 0. For each $x \in \cup \mathcal{U}$ we let
\[
r(x) = \sup \nu (x)
\]
and note that $0 < r(x) \leq \infty$. We let
\[
l = \inf \{r(x) : x \in X\}.
\]
We call $l$ the **Lebesgue radius** of $\mathcal{U}$. Evidently,
\[
0 < r < l \iff \text{ for each } a \in X \text{ there is } U \in \mathcal{U} \text{ such that } U^a(r) \subset U.
\]
Lemma 3.1. Suppose $\mathcal{U}$ is an open covering of $X$ and $0 < s < \infty$. Then $\{x \in X : \rho_{\mathcal{U}}(x) > s\}$ is open.

Proof. Let $G = \{x \in X : \rho_{\mathcal{U}}(x) > s\}$ and suppose $a \in G$. Choose $t, u$ such that $s < t < u < \rho_{\mathcal{U}}(a)$. Next, choose $U \in \mathcal{U}$ such that $U^{a}(u) \subset U$.

Suppose $x \in U^{a}(u-t)$. Then

$$U^{x}(t) \subset U^{a}(u) \subset U$$

so $\rho_{\mathcal{U}}(x) \geq t > s$. That is, $U^{x}(u-t) \subset G$ so $G$ is open. □

Theorem 3.1. Suppose $X$ is compact and $\mathcal{U}$ is an open covering of $X$. Then $l_{\mathcal{U}} > 0$.

Proof. Let

$$W = \{\{x \in X : \rho_{\mathcal{U}}(x) > s\} : 0 < s < \infty\}.$$ 

From the Lemma we infer that $W$ is an open covering of $X$. Since $X$ is compact there is a finite subfamily of $W$ whose union contains $X$. Since $W$ is nested we infer that some member of $W$ contains $X$; that is, there is $s$ such that $0 < s < \infty$ and $X \subset \{x \in X : \rho_{\mathcal{U}}(x) > s\}$; we have $s \leq l_{\mathcal{U}}$ for any such $s$. □

4. Uniform continuity.

Definition 4.1. Suppose $(Y, \sigma)$ is a metric space, $A \subset X$ and $f : A \to Y$. We say $f$ is uniformly continuous if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$a, x \in A \text{ and } \rho(x, a) < \delta \Rightarrow \sigma(f(x), f(a)) < \epsilon.$$ 

Theorem 4.1. Suppose $(X, \rho)$ and $(Y, \sigma)$ are metric spaces, $X$ is compact, $f : X \to Y$

and $f$ is continuous.

Then $f$ is uniformly continuous.

Proof. Let $\epsilon > 0$. Let

$$\mathcal{U} = \{U : U \text{ is an open subset of } X \text{ and } \text{diam} f[U] < \epsilon\}.$$ 

Suppose $a \in X$. Since $f$ is continuous at $a$ we may choose $\eta > 0$ such that $f[\mathcal{U}^{a}(\eta)] \subset f^{a}(\epsilon/2)$. Thus, with $U = \mathcal{U}^{a}(\eta)$, $\text{diam} f[U] < \epsilon$ so $\mathcal{U}$ is an open covering of $X$.

Since $X$ is compact $l_{\mathcal{U}}$ is positive so we may choose $\delta$ such that $0 < \delta < l_{\mathcal{U}}$.

Suppose $a \in X$. There is $U \in \mathcal{U}$ such that $\mathcal{U}^{a}(\delta) \subset U$. Thus

$$\text{diam} f[\mathcal{U}^{a}(\delta)] \leq \text{diam} f[U] < \epsilon.$$ 

Thus

$$x \in A \text{ and } \rho(x, a) < \delta \Rightarrow \sigma(f(x), f(a)) \leq \text{diam} f[U] < \epsilon.$$ 

□
5. **Total boundedness.**

**Definition 5.1.** $X$ is **totally bounded** if for each $\epsilon > 0$ there is a finite family $\mathcal{F}$ of subsets of $X$ such that

1. $X = \bigcup \mathcal{F}$

and

2. $\text{diam } F \leq \epsilon$ whenever $F \in \mathcal{F}$.

**Theorem 5.1.** $X$ is compact if and only if it is complete and totally bounded.

*Proof.* We leave as an exercise for the reader the straightforward verification that if $X$ is compact then $X$ is complete and totally bounded.

Suppose $X$ is complete and totally bounded and let $\mathcal{U}$ be an open covering of $X$. Call a subset $A$ of $X$ **good** if there is a finite subfamily of $\mathcal{U}$ whose union contains $A$ and call a subset $A$ of $X$ **bad** if it is not good. Note that the union of a finite family of good sets is good. We want show that $X$ is good.

Suppose $X$ were bad. Let $r_1, r_2, \ldots$ be a sequence of positive real numbers with limit zero. For each $i = 1, 2, \ldots$ let $F_i$ be a finite subset of $X$ such that

$$X = \bigcup \{B^{r_i}(x_i) : x_i \in F_i\};$$

such sets exist because $X$ is totally bounded. There would be $x_1 \in F_1$ such that $B^{r_1}(x_1)$ is bad; otherwise $X$ would be the union of the finite family $\{B^{r_1}(x_1) : x_1 \in F_1\}$ of good sets and would therefore be good. There would be $x_2 \in F_2$ such that $B^{r_2}(x_1) \cap B^{r_2}(x_2)$ is bad; otherwise $B^{r_2}(x_1)$ would be the union of the family $\{B^{r_2}(x_1) \cap B^{r_2}(x_2) : x_2 \in F_2\}$ of good sets and would therefore be good. Continuing in this way we would obtain a sequence $x_1, x_2, \ldots$ in $X$ such that

$$C_m = \bigcap_{i=1}^{m} B_{r_i}(x_i)$$

would be bad.

These sets would be nonempty since the empty set is good. By the completeness of $X$ there would be a point $c \in \bigcap_{m=1}^{\infty} C_m$. But $c \in U$ for some $U \in \mathcal{U}$ and, since $\text{diam } C_m$ tends to zero as $m$ tends to infinity, we would have $C_m \subset U$ for sufficiently large $m$. For these $m$, $C_m$ would be good. \hfill $\Box$

**Corollary 5.1.** A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

*Proof.* Suppose $A$ is a compact subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff, $A$ is closed by virtue of a previous Theorem. Moreover, $\{U^0(r) : 0 < r < \infty\}$ is an open covering of $\mathbb{R}^n$ and therefore $A$; since $A$ is compact, it has a finite subfamily whose union contains $A$. It follows that $A \subset U^0(r)$ for some positive real number $r$ so $A$ is bounded.

Suppose $A$ is a closed and bounded subset of $\mathbb{R}^n$. It follows easily from the fact that $\mathbb{R}^n$ is complete and $A$ is closed that $A$, considered as a metric space, is complete. $A$ is totally bounded as well; in fact, for any positive real number $\epsilon$ the set $A$ is contained in the union of a finite subfamily of the family

$$\{B^{\epsilon z}(\sqrt{n} \epsilon) : z \in \mathbb{Z}^n\}$$

because $A$ is bounded. It now follows from the previous Theorem that $A$ is compact. \hfill $\Box$
6. LIPSCHITZ CONSTANTS.

Suppose \((Y, \sigma)\) is a metric space, \(A \subset X\) and 
\[ f : A \to Y. \]

**Proposition 6.1.** \(f\) is continuous if and only if for each \(a \in X\) and each \(\epsilon \in (0, \infty)\) there is \(\delta \in (0, \infty)\) such that 
\[ f[U^a(\delta)] \subset U^{f(a)}(\epsilon). \]

**Proof.** Proceed as we did in the case when \(X = \mathbb{R}^n\) and \(Y = \mathbb{R}^m\). □

**Definition 6.1.** We let 
\[ \text{Lip}(f) \]
be the infimum of the set of \(M \in [0, \infty)\) such that 
\[ \sigma(f(x), f(a)) \leq M\rho(x, a) \text{ whenever } x, a \in X. \]

Note that (1) holds with \(M = \text{Lip}(f)\). We call this extended real number the **Lipschitz constant of** \(f\). We say \(f\) is **Lipschitzian** if \(\text{Lip}(f) < \infty\). We say \(f\) is **locally Lipschitzian** if \(\text{Lip}(f|B) < \infty\) whenever \(B\) is a bounded subset of \(X\).

Note that 
\[ \text{diam} f[B] \leq \text{Lip}(f) \text{diam } B \quad \text{whenever } B \subset X. \]

**Proposition 6.2.** If \(f\) is locally Lipschitzian then \(f\) is continuous.

**Proof.** This follows directly from (2). □

**Theorem 6.1.** Suppose \(Y\) is complete and \(\text{Lip}(f) < \infty\). Then \(f\) has a unique continuous extension to \(\text{cl } A\) and the Lipschitz constant of this extension equals the Lipschitz constant of \(f\).

**Proof.** Let \(F\) be the set of \((a, b) \in (\text{cl } A) \times Y\) such that 
\[ b \in \bigcap_{0 < \delta < \infty} \text{cl } f[U^a(\delta)]. \]

Since 
\[ f[U^a(\delta)] \neq \emptyset \]
and 
\[ \text{diam } f[U^a(\delta)] \leq \text{Lip}(f) \text{diam } U^a(\delta) \leq 2\delta \quad \text{for any } a \in \text{cl } A \]
and since \(Y\) is complete we infer that \(F\) is a function whose domain is the closure of \(A\). Since 
\[ f(a) \in \bigcap_{0 < \delta < \infty} \text{cl } f[U^a(\delta)] \]
for any \(a \in A\) we find that \(F|A = f\).

Suppose \(c_i \in \text{cl } A, \ i = 1, 2,\) and let \(r\) and \(s\) be positive real numbers. Since \(F(c_i) \in \text{cl } f[U^{c_i}(r)]\) we may choose \(a_i \in U^{c_i}(r)\) such that \(\sigma(F(c_i), f(a_i)) < s, \ i = 1, 2\). Then
\[ \sigma(F(c_1), F(c_2)) \leq \sigma(F(c_1), f(a_1)) + \sigma(f(a_1), f(a_2)) + \sigma(f(a_2), F(c_2)) \]
\[ \leq s + \text{Lip}(f)\rho(a_1, a_2) + s \]
\[ \leq s + \text{Lip}(f)(\rho(c_1, c_1) + \rho(c_1, c_2) + \rho(c_2, a_2)) + s \]
\[ = 2s + \text{Lip}(f)(2r + \rho(c_1, c_2)); \]
owing to the arbitrariness of \( r \) and \( s \) we infer that
\[
\sigma(F(c_1), F(c_2)) \leq \text{Lip}(f) \rho(c_1, c_2).
\]
it follows that \( \text{Lip}(F) \leq \text{Lip}(f) \).

Finally, suppose that \( g : \text{cl}A \to Y \) is continuous, \( g|A = f \) and \( c \in \text{cl}A \). Let \( \epsilon > 0 \). Then there is \( \delta > 0 \) such that
\[
x \in B^c(\delta) \cap \text{cl}A \Rightarrow g(x) \in B^g(c)(\epsilon).
\]
Let \( a \in A \cap B^c(\min\{\delta, \epsilon\}) \); such an \( a \) exists because \( c \in \text{cl}A \). Then, since \( g(a) = f(a) = F(a) \) we have
\[
\sigma(g(c), F(c)) \leq \sigma(g(c), g(a)) + \sigma(F(a), F(c)) \leq \epsilon + \text{Lip}(F)\epsilon.
\]
Owing to the arbitrariness of \( \epsilon \) we infer that \( g(c) = F(c) \). Thus \( g = F \). \( \square \)