Submanifolds.

Let $n$ be a positive integer.

**Definition.** We say $f$ is an $n$-diffeomorphism if

1. $f$ is a function whose domain and range of $f$ are open subsets of $\mathbb{R}^n$;
2. $f$ is smooth;
3. $f$ is univalent and, for each $x \in \text{dom} f$, $\partial f(x)$ carries $\mathbb{R}^n$ isomorphically onto itself.

Whenever $U$ and $V$ are open subsets of $\mathbb{R}^n$ we let $\text{Diffeo}_n$ be the set of ordered triples $(U, F, V)$ such that $F$ is an $n$-diffeomorphism with domain $U$ and range $V$.

**Proposition.** We have

1. $\emptyset$ is an $n$-diffeomorphism;
2. if $F$ is an $n$-diffeomorphism and $W$ is an open subset of $\mathbb{R}^n$ then $F|W$ is an $n$-diffeomorphism;
3. if $U$ is a family of open subsets of $\mathbb{R}^n$, $F : \bigcup U \to \mathbb{R}^n$, $F$ is univalent and $F|U$ is a $n$-diffeomorphism for each $U \in \mathcal{U}$ then $F$ is an $n$-diffeomorphism;
4. if $F$ is an $n$-diffeomorphism then $F^{-1}$ is an $n$-diffeomorphism;
5. if $F, G$ are $n$-diffeomorphisms then $F \circ G$ is an $n$-diffeomorphism.

**Proof.** Exercise for the reader. It will be necessary to use the Inverse Function Theorem and its Corollaries, the Chain Rule and the fact the inversion on $\text{GL}(\mathbb{R}^n)$ is smooth. □

Suppose $m$ is an integer and $0 \leq m \leq n$.

**Definition.**

$\mathbb{R}^{m,n} = \{ x \in \mathbb{R}^n : x_i = 0 \text{ whenever } i < m \leq n \}$.

Let

$p_{m,n}, \ q_{m,n}, \ i_{m,n}, \ j_{m,n}$

be defined by the following requirements:

$p_{m,n} : \mathbb{R}^n \to \mathbb{R}^m$,
$q_{m,n} : \mathbb{R}^n \to \mathbb{R}^{n-m}$,
$i_{m,n} : \mathbb{R}^m \to \mathbb{R}^{m,n}$,
$j_{m,n} : \mathbb{R}^{n-m} \to (\mathbb{R}^{m,n})^\perp$;

if $m = 0$ then

$p_{m,n} = 0$ and $q_{m,n} = i_{\mathbb{R}^n}$;

if $1 < m < n$ then

$p_{m,n}(x) = \sum_{i=1}^m x_i e_i, \ x \in \mathbb{R}^m$ and $q_{m,n}(y) = \sum_{j=1}^{n-m} y_j e_{m+j}, \ y \in \mathbb{R}^{n-m}$;

if $m = n$ then

$p_{m,n} = i_{\mathbb{R}^n}$ and $q_{m,n} = 0$;

and

$i_{\mathbb{R}^n} = i_{m,n} \circ p_{m,n} + j_{m,n} \circ q_{m,n}$.

We let

$\mathcal{U}^n = \{ x \in \mathbb{R}^n : |x| < 1 \}$
and we let
\[ U_{m,n} = U^m \cap R^{m,n}. \]
Whenever \( m \geq 1 \) we let
\[ U_{m,n,+} = \{ x \in U_{m,n} : x_m > 0 \}. \]

**Definition.** Suppose \( V \) is an open subset of \( R^n \). We let
\[ M_m(V) \]
be the family of nonempty subsets \( M \) of \( V \) such that
1. if \( a \in M \) there is \( (U^n, \Phi, U) \in \text{Diffeo}_n \) such that \( a \in U \subset V \), \( \Phi(0) = a \) and \( U \cap M = \Phi[U_{m,n}] \).
2. if \( m \geq 1 \) and \( b \in (V \sim \text{cl} M) \sim M \) there is \( (U^n, \Phi, U) \in \text{Diffeo}_n \) such that \( b \in U \subset V \), \( \Phi(0) = b \) and \( U \cap M = \Phi[U_{m,n,+}] \).

We call the members of \( M_m(V) \) smooth \( m \)-dimensional submanifolds of \( V \).
For each \( M \in M_{m,n}(V) \) we set
\[ \partial M = (V \cap \text{cl} M) \sim M. \]

**Theorem.** Suppose \( V \) is an open subset of \( R^n \) and \( M \) is a nonempty subset of \( V \). Then
1. \( M \in M_0(V) \) if and only if \( M \) is a nonempty subset of \( V \) which meets any compact subset of \( V \) in a finite set.
2. if \( M \in M_0(V) \) then \( \partial M = \emptyset \).
3. if \( M \in M_0(V) \) then \( \partial M = \emptyset \) if and only if each connected component of \( M \) is a connected component of \( V \).
4. if \( m \geq 1 \) and \( M \in M_m(M) \) then \( \partial M \in M_{m-1}(V) \) and \( \partial(\partial M) = \emptyset \).

**Proof.** These are straightforward consequences of the definitions. \( \square \)

**Theorem.** Suppose \( 1 \leq m < n \), \( V \) is an open subset of \( R^n \) and \( M \) is a nonempty subset of \( V \). Then
1. for each \( a \in M \) there are an open subset \( U \) of \( V \) and a smooth map \( F : U \to R^{n-m} \) such that
\[ \dim \text{rng} \partial f(a) = n - m \]
and
\[ M \cap U = \{ x \in V : F(x) = F(a) \}; \]
2. for each \( b \in (V \cap \text{cl} M) \sim M \) there are an open subset \( U \) of \( V \) and smooth maps
\[ F : U \to R^{n-m} \quad \text{and} \quad g : U \to R \]
such that
\[ \dim \text{rng} \partial f(a) = n - m, \quad \partial g(a) \notin \text{span} \{ \partial F^i(a) : i = 1, \ldots, n-m \} \]
and
\[ U \cap M = \{ x \in U : F(x) = F(a) \text{ and } g(x) > g(a) \}. \]

**Remark.** Note that if (2) holds there is an open subset \( T \) of \( U \) such that \( a \in T \) and
\[ T \cap \text{cl} M = \{ x \in T : F(x) = F(a) \text{ and } g(x) \geq g(a) \} \]
and
\[ T \cap \partial M = \{ x \in T : F(x) = F(a) \text{ and } g(x) = g(a) \}. \]

**Proof.** Exercise for the reader. Use the Implicit Function Theorem. \( \square \)
**Theorem.** Suppose $V$ is an open subset of $\mathbb{R}^n$ and $M$ is a nonempty subset of $V$. Then $M \in M_n(V)$ if and only if for each $b \in (V \cap \text{cl} M) \sim M$ there are an open subset $U$ of $V$ and a smooth map

$$g : U \to \mathbb{R}$$

such that

$$\partial g(a) \neq 0$$

and

$$U \cap \text{cl} M = \{x \in U : g(x) > g(a)\}.$$

**Remark.** Note that if $g$ is as above then there is an open subset $T$ of $U$ such that $a \in T$ and

$$T \cap \partial M = \{x \in T : g(x) = g(a)\}.$$

**Proof.** Exercise for the reader. Use the Implicit Function Theorem. □

**Immersions.**

**Definition.** Suppose $T$ is an open subset of $\mathbb{R}^m$ and $V$ is an open subset of $\mathbb{R}^n$. By a proper immersion of $T$ into $V$ we mean a smooth univalent map $\phi : T \to V$ such that

$$\dim \text{rng} \partial \phi(t) = m \quad \text{whenever } t \in T$$

and

$$\phi^{-1}[K] \text{ is a compact subset of } T \text{ whenever } K \text{ is a compact subset of } V.$$  

We let

$$\text{Imm}_{m,n}$$

be the set of ordered triples $(T, \phi, V)$ such that $T$ is an open subset of $\mathbb{R}^m$, $V$ is an open subset of $\mathbb{R}^n$ and $\phi$ is a proper immersions of $T$ into $V$.

**Theorem.** Suppose $T$ is an open subset of $\mathbb{R}^m$, $V$ is an open subset of $\mathbb{R}^n$ and $\phi : T \to V$ is a smooth univalent map such that

$$\dim \text{rng} \partial \phi(t) = m \quad \text{whenever } t \in T.$$  

Let $M = \text{rng} \phi$. Then the following conditions are equivalent:

(1) 

$$(T, \phi, U) \in \text{Imm}_{m}.$$  

(2) 

$$M \in M_{m}(V) \text{ and } \partial M = \emptyset.$$  

**Proof.** Let $M = \text{rng} \phi$.

**Part One.** Suppose (1) holds and $a \in V \cap \text{cl} M$. Let

$$\mathcal{K} = \{\phi^{-1}[B_a(r)] : 0 < r < \infty \text{ and } B_a(r) \subset V\}$$

Then $\mathcal{K}$ is a nested family of nonempty compact subsets of $T$ any point of whose nonvoid intersection is carried to $a$ by $\phi$. Since $\phi$ is univalent there is $c \in T$ such that $\bigcap \mathcal{K} = \{c\}$ and $\phi(c) = a$. Thus

$$\text{for any open subset } S \text{ of } T \text{ such that } c \in S \text{ there is } r > 0 \text{ such that } \phi^{-1}[U_a(r)] \subset S.$$  

In particular,

$$V \cap \text{cl} M = V \cap M.$$  

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Choose $l \in \otimes (\mathbb{R}^{n-m}, \mathbb{R}^n)$ such that

$$\text{rng } \partial \phi (c) + \text{rng } l = \mathbb{R}^n.$$  

and let

$$G(t, u) = \phi(t) + l(u), \ (t, u) \in T \times \mathbb{R}^{n-m}.$$  

Since $\text{rng } \partial G(c, 0) = \mathbb{R}^n$ we may apply the Inverse Function Theorem to obtain open subsets $S$ of $T$ and $W$ of $\mathbb{R}^{n-m}$ such that $(c, 0) \in S \times W$ and $H = G|(S \times W)$ is an $m$-diffeomorphism. Now choose $r > 0$ such that if $U = U_a(r)$ then

$$U \subset \text{rng } H \text{ and } \phi^{-1}[U] \subset S.$$  

Let $q(t, u) = u$ for $(t, u) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and set

$$F = (q \circ H^{-1}) |U.$$  

Since $F(H(t, u)) = u$ for whenever $(t, u) \in S \times W$ and $H(t, u) \in U$ we find that

$$\text{rng } \partial F(c) = n - m.$$  

Suppose $x \in M \cap U$. Then $x = \phi(t)$ for some $t \in S$. Since $H(t, 0) = x$ we find that $F(x) = 0$. Thus

$$\{x \in U : F(x) = F(a)\} = M \cap U$$  

and (2) holds.

**Part Two.** Suppose (2) holds and $a \in M$. It will suffice to show that there is $r > 0$ such that $\phi^{-1}[B_a(r)]$ is a compact subset of $T$. Let $(U^n, \Phi, U) \in \text{Diffeo}_n$ be such that $a \in U \subset V$

$$U \cap M = \Phi[U^{m,n}].$$

Set $S = \phi^{-1}[U]$ and set $\psi = (p_{m,n} \circ \Phi^{-1} \circ \phi)|S$. Note that

$$\text{rng } \partial \psi(t) = \mathbb{R}^m, \ t \in S.$$  

Thus $(S, \psi, \psi[S])$ is an $m$-diffeomorphism by earlier results. It follows that $\psi^{-1}[L]$ is a compact subset of $S$ whenever $L$ is a compact subset of $\psi[S]$. Choose $r > 0$ such that $B_a(r) \subset U$. Then $L = p_{m,n} \circ \Phi^{-1}[B_a(r)]$ is a compact subset of $\psi[S]$ and

$$\phi^{-1}[B_a(r)] = \psi^{-1}[L]$$  

so $\phi^{-1}[B_a(r)]$ is a compact subset of $S$, as desired. □

**Theorem.** Suppose

$$(T_i, \phi_i, V_i) \in \text{Imm}_m, \ i = 1, 2$$

and

$$V_2 \cap \text{rng } \phi_1 = V_1 \cap \text{rng } \phi_2.$$  

Then

$$(\phi^{-1}_1[V_2], \phi^{-1}_2 \circ \phi_1, \phi^{-1}_2[V_1]) \in \text{Diffeo}_m.$$  

**Proof.** Suppose $c_1 \in \phi^{-1}[V_2]$. Since $\text{rng } \phi_1 \in M_m(V_1)$ we may choose $(U^n, \Phi, U) \in \text{Diffeo}_n$, such that $\phi_1(t_1) \in U \subset V_1 \cap V_2$ and such that $U \cap \text{rng } \phi_1 = \Phi[U^{m,n}]$. Let

$$\psi_i = p_{m,n} \circ \Phi^{-1} \circ \phi_i, i = 1, 2.$$  

Evidently, $\psi_i$ is a smooth univalent map carrying the open subset $\phi^{-1}_i[U_i]$ of $\mathbb{R}^m$ onto the open subset $(p_{m,n} \circ \Phi^{-1})[U]$ of $\mathbb{R}^{m}$, $i = 1, 2$ and

$$(\phi^{-1}_2 \circ \phi_1)\phi^{-1}_1[U] = \psi^{-1}_2 \circ \psi_1.$$
Since $c_1 \in \phi_1^{-1}[U]$ the proof will be complete if we can show that

$$ (\phi_1^{-1}[U], \psi_i, (p_{m,n} \circ \Phi^{-1})(U)) \in \text{Diffeo}_m, \ i = 1, 2, $$

and this will follow if we can show that

$$ \text{rng} \partial \psi_i(t_i) = \mathbb{R}^m, \ t_i \in \phi_i^{-1}[U], \ i = 1, 2. $$

So suppose $i \in \{1, 2\}$ and $t_i \in \phi_i^{-1}[U]$. Then $\dim \text{rng} \partial(\Phi^{-1} \circ \phi_i)(t) = m$ by the Chain Rule. But as the range of $\Phi^{-1} \circ \phi_i$ is a subset of $\mathbb{R}^{m,n}$ we find that

$$ \text{rng} \psi_i = \text{rng} \partial(p_{m,n} \circ \Phi^{-1} \circ \phi_i)(t_i) = p_{m,n}[\text{rng} \partial(\Phi^{-1} \circ \phi_i)(t_i)] = \mathbb{R}^m. $$

\[ \square \]

**Definition.** Suppose $V$ is an open subset of $\mathbb{R}^n$ and $M \in M_m(V)$. We say $\phi$ is a local parameter for $M$ if there are $T$ and $U$ such that $U \subset V$, $(T, \phi, U) \in \text{Imm}_m$ and

$$ U \cap M = \text{rng} \phi. $$

We have just shown that if $\phi_i, \ i = 1, 2$ are local parameters for $M$ then $\phi_2^{-1} \circ \phi_1 \in \text{Diffeo}_m$.  

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