Integration of a scalar function over a submanifold.

Suppose n is a positive integer, V is an n-dimensional inner product space, $0 \le m < n$ and $M \in \mathbf{M}_m(V)$.

Definition. We say (T, ϕ) is a **smooth local parameter for** M if

(1) T is an open subset of $\mathbf{R}^m$;

(2) $\phi : T \to V$ is smooth;

(3) $\mathbf{rng}\, \phi \subset M$;

(4) ϕ is univalent;

(5) $\mathbf{dim}\, \mathbf{rng}\, \partial\phi(t) = m$ whenever $t \in T$.

If (T, ϕ) is a smooth local parameter for M we define the smooth function

$$J_m\phi : T \to (0, \infty),$$

the m-**Jacobian of** ϕ, by letting

$$J_m\phi(t) = \sqrt{\mathbf{det}\, \partial\phi(t)^* \circ \partial\phi(t)} \quad \text{whenever } t \in T.$$

We say a function f on M with values in a topological space is **Lebesgue measurable** if $f \circ \phi$ is Lebesgue measurable whenever (T, ϕ) is a local coordinate for M.

We let

$$\mathcal{L}_M^+$$

be the family of Lebesgue measurable functions.

Remark. Suppose $a \in M$. Then there are an open subset U of V and $(U, \Phi, \mathbf{U}^n)$ such that $a \in U$, $\Phi(a) = 0$ and $\Phi[M \cap U] = \mathbf{U}^{m,n}$. It follows that $(\mathbf{U}^{m,n}, \Phi^{-1} \circ i_{m,n})$ is a local parameter for M whose range contains a.

We say (W, ψ) is a **smooth local coordinate for** M if

(1) W is a subset of M which is open relative to V;

(2) $\psi : W \to \mathbf{R}^m$;

(3) $(\psi[W], \psi^{-1})$ is a smooth local parameter for M.

Theorem. Suppose

(1) (T, ϕ) is a smooth local parameter for M;

(2) U is an open subset of V and $\phi(t) \in U$;

(3) $(U, \Phi, \mathbf{U}^n) \in \mathbf{Diffeo}_n$, $\Phi(\phi(t)) = 0$ and $\Phi[M \cap U] = \mathbf{U}^{m,n}$.

Then

$$p_{m,n} \circ \Phi \circ \phi \in \mathbf{Diffeo}_m. \tag{4}$$

Proof. Let $F = p_{m,n} \circ \Phi \circ \phi$. Then $\mathbf{dmn}\, F = T \cap \phi^{-1}[U]$ which is an open subset of $\mathbf{R}^m$, F is smooth and F is univalent. Moreover, by the Chain Rule, we have that $\mathbf{rng}\, \partial f(u) = \mathbf{R}^m$ whenever $u \in \mathbf{dmn}\, F$. The assertion to be proved now follows from a Corollary to the Inverse Function Theorem. □

Corollary. Suppose (U, ψ) is a smooth local coordinate for M. then $(\psi[U], \psi^{-1})$ is a smooth local parameter for M.
Proof. We have only to verify that $\psi[U]$ is open relative to M and this follows directly from the preceding Proposition. $\square$

Corollary. Suppose (T_i, ϕ_i), $i = 1, 2$, are smooth local parameters for M

$$(T_1 \cap \phi_1^{-1} \circ \phi_2[T_2], \phi_2^{-1} \circ \phi_1, T_2 \cap \phi_2^{-1} \circ \phi_1[T_1]) \in \mathbf{Diffeo}_m.$$

Proof. It is evident that the $T_1 \cap \phi_1^{-1}[T_2]$ and $T_2 \cap \phi_2^{-1}[T_2]$ are open sets of $\mathbf{R}^m$ which are the domain and range of $\phi_2^{-1} \circ \phi_1$. It is evident that $\phi_2^{-1} \circ \phi_1$ is a univalent function.

Suppose $t_1 \in T_1 \cap \phi_1^{-1} \circ \phi_2[T_2]$. Let $(U, \Phi, \mathbf{U}^n)$ be such that U is an open subsets of V, $\phi_1(t_1) \in U$, $(U, \Phi, \mathbf{U}^n) \in \mathbf{Diffeo}_n$, $\Phi(\phi_1(t_1)) = 0$ and $\Phi[M \cap U] = \mathbf{U}^{m,n}$. We have shown above that

$$p_{m,n} \circ \Phi \circ \phi_i \in \mathbf{Diffeo}_m, \ i = 1, 2.$$

Since

$$(\phi_2^{-1} \circ \phi_1)|\phi_1^{-1}[U \cap \phi_2[T_2]] = (p_{m,n} \circ \Phi \circ \phi_2)^{-1} \circ (p_{m,n} \circ \Phi \circ \phi_1)$$

and since $\phi_1^{-1}[U \cap \phi_2[T_2]]$ is an open subset of $\mathbf{R}^m$ containing t_1 the proof is complete. $\square$

Lemma. Suppose (T_i, ϕ_i), $i = 1, 2$, are local parameters for M, $f \in \mathcal{L}_M^+$ and

$$f(a) = 0 \quad \text{whenever } a \in \phi_1[T_1] \cap \phi_1[T_2].$$

Then

$$\int_{T_1} f \circ \phi_1 \, J_m \phi_1 = \int_{T_2} f \circ \phi_2 \, J_m \phi_2.$$

Proof. Replacing T_1 with $\phi_1^{-1}[T_2]$ and T_2 by $\phi_2^{-1}[T_1]$ we may assume that $\phi_1[T_1] = \phi_2[T_2]$. From the preceding work we have that $(T_1, \phi_2^{-1} \circ \phi_1, T_2) \in \mathbf{Diffeo}_m$. From the Change of Variables Formula for Multiple Intgrals we infer that

$$\int_{T_2} f \circ \phi_2 \, J_m \phi_2 = \int_{T_1} f \circ \phi_2 \circ (\phi_2^{-1} \circ \phi_1) \, J_m \phi_2 \circ (\phi_2^{-1} \circ \phi_1) \, |\mathbf{det}\, \partial(\phi_2^{-1} \circ \phi_1)|.$$

The Lemma will follow if we can show that

$$J_m \phi_1 = (J_m \phi_2) \circ (\phi_2^{-1} \circ \phi_1) \, |\mathbf{det}\, \partial(\phi_2^{-1} \circ \phi_1).$$

Suppose $t_1 \in T_1$ and let $t_2 = \phi_2^{-1} \circ \phi_1(t_1) \in T_2$. By the Chain Rule we have

$$\partial \phi_1(t_1) = \partial \phi_2(t_2) \circ \partial(\phi_2^{-1} \circ \phi_1)(t_1)$$

so that

$$\partial \phi_1(t_1)^* = \partial(\phi_2^{-1} \circ \phi_1)(t_1)^* \circ \partial \phi_2(t_2)^*.$$

Thus

$$\begin{aligned} &\partial \phi_1(t_1)^* \circ \partial \phi_1(t_1) \\ &\qquad = \Big(\partial(\phi_2^{-1} \circ \phi_1)(t_1)^* \circ \partial \phi_2(t_2)^*\Big) \circ \Big(\partial \phi_2(t_2) \circ \partial(\phi_2^{-1} \circ \phi_1)(t_1)\Big) \\ &\qquad = \partial(\phi_2^{-1} \circ \phi_1)(t_1)^* \circ \Big(\partial \phi_2(t_2)^* \circ \partial \phi_2(t_2)\Big) \circ \partial(\phi_2^{-1} \circ \phi_1)(t_1) \end{aligned}$$

so that, by the Product Rule for Determinants and the fact that

$$\mathbf{det}\, L^* = \mathbf{det}\, L \quad \text{whenever } L \in \mathbf{L}(\mathbf{R}^m; \mathbf{R}^m)$$

we find that

$$J_m\phi_1(t_1) = |\mathbf{det}\partial(\phi_2^{-1} \circ \phi_1)(t_1)|\, J_m\phi_2(t_2).$$

□

Lemma. There is a countable family $\mathcal{T}$ of smooth local parameters for M such that

$$M = \cup\{T : (T, \phi) \in \mathcal{T}\}.$$

Proof. Exercise for the reader. □

Lemma. Suppose $f \in \mathcal{L}_M^+$ and

(1) (S_μ, α), $\mu = 1, 2, 3, \ldots$, are smooth local parameters for M;

(2) $\cup_{\mu=1}^{\infty} \mathbf{rng}\, \alpha_\mu = M$;

(3) $A_1 = \mathbf{rng}\, \alpha_1$ and $A_\mu = \mathbf{rng}\, \alpha_\nu \sim \cup_{\gamma<\mu} \mathbf{rng}\, \alpha_\gamma$ if $mu > 1$;

(4)

$$I = \sum_{\mu=1}^{\infty} \int_{S_\mu} (f\, 1_{A_\mu}) \circ \alpha_\mu J_m \alpha_\mu;$$

(5)(T_ν, β_ν), $\nu = 1, 2, 3, \ldots$, are smooth local parameters for M;

(6) $\cup_{\nu=1}^{\infty} \mathbf{rng}\, \beta_\nu = M$;

(7) $B_1 = \mathbf{rng}\, \beta_1$ and $B_\nu = \mathbf{rng}\, \beta_\nu \sim \cup_{\delta<\nu} \mathbf{rng}\, \beta_\delta$;

(8)

$$J = \sum_{\nu=1}^{\infty} \int_{T_\nu} (f\, 1_{B_\nu}) \circ \beta_\nu J_m \beta_\nu.$$

Then

$$I = J.$$

Proof. We have

$$1_M = \sum_{\mu=1}^{\infty} 1_{A_\mu} \quad \text{and} \quad 1_N = \sum_{\nu=1}^{\infty} 1_{B_\nu}.$$

Thus

$$I = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \int_{S_\mu} (f\, 1_{A_\mu} 1_{B_\nu})\, J_m \alpha_\mu \quad \text{and} \quad J = \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\infty} \int_{T_\nu} (f\, 1_{A_\mu} 1_{B_\nu}) \circ \beta_\nu J_m \beta_\nu.$$

(Why?) That $I = J$ now follows from a previous Proposition. □

Theorem. There is one and only one function

$$I_M^+ : \mathcal{L}_M^+ \to [0, \infty]$$

such that

(1) $I_M^+(\sum_{\nu=0}^{\infty} f_\nu) = \sum_{\nu=0}^{\infty} I_M^+(f_\nu)$ whenever f is a sequence in $\mathcal{L}_M^+$;

(2) $I_M^+(f) = \int_T f \circ \phi\, J_m \phi$ whenever $f \in \mathcal{M}$ and $f(a) = 0$ whenever $a \notin \mathbf{rng}\, \phi$.

Proof. Combine the preceding two Lemmas. □

Definition. We let

$$\mathcal{L}_M$$

be the vector space of Lebesgue measurable function $f : M \to \mathbf{R}$ such that $I_M^+(|f|) < \infty$. For each $f \in \mathcal{L}_M$ we let

$$I_M(f) = I_M^+(f^+) - I_M^+(f^-).$$