Integration of a scalar function over a submanifold.

Suppose $n$ is a positive integer, $V$ is an $n$-dimensional inner product space, $0 \leq m < n$ and $M \in \mathbf{M}_m(V)$.

**Definition.** We say $(T, \phi)$ is a smooth local parameter for $M$ if

1. $T$ is an open subset of $\mathbb{R}^m$;
2. $\phi : T \to V$ is smooth;
3. $\operatorname{rng} \phi \subseteq M$;
4. $\phi$ is univalent;
5. $\dim \operatorname{rng} \partial \phi(t) = m$ whenever $t \in T$.

If $(T, \phi)$ is a smooth local parameter for $M$ we define the smooth function

$$J_m \phi : T \to (0, \infty),$$

the $m$-Jacobian of $\phi$, by letting

$$J_m \phi(t) = \sqrt{\det \partial \phi(t)^* \circ \partial \phi(t)} \quad \text{whenever } t \in T.$$

We say a function $f$ on $M$ with values in a topological space is **Lebesgue measurable** if $f \circ \phi$ is Lebesgue measurable whenever $(T, \phi)$ is a local coordinate for $M$.

We let $\mathcal{L}^+_M$ be the family of Lebesgue measurable functions.

**Remark.** Suppose $a \in M$. Then there are an open subset $U$ of $V$ and $(U, \Phi, U^n)$ such that $a \in U$, $\Phi(a) = 0$ and $\Phi[M \cap U] = U^{m,n}$. It follows that $(U^{m,n}, \Phi^{-1} \circ i_{m,n})$ is a local parameter for $M$ whose range contains $a$.

We say $(W, \psi)$ is a smooth local coordinate for $M$ if

1. $W$ is a subset of $M$ which is open relative to $V$;
2. $\psi : W \to \mathbb{R}^m$;
3. $(\psi[W], \psi^{-1})$ is a smooth local parameter for $M$.

**Theorem.** Suppose

1. $(T, \phi)$ is a smooth local parameter for $M$;
2. $U$ is an open subset of $V$ and $\phi(t) \in U$;
3. $(U, \Phi, U^n) \in \mathbf{Diffeo}_m$, $\Phi(\phi(t)) = 0$ and $\Phi[M \cap U] = U^{m,n}$.

Then

$$p_{m,n} \circ \Phi \circ \phi \in \mathbf{Diffeo}_m.$$

**Proof.** Let $F = p_{m,n} \circ \Phi \circ \phi$. Then $\mathbf{dmn} F = T \cap \phi^{-1}(U)$ which is an open subset of $\mathbb{R}^m$, $F$ is smooth and $F$ is univalent. Moreover, by the Chain Rule, we have that $\operatorname{rng} \partial f(u) = \mathbb{R}^m$ whenever $u \in \mathbf{dmn} F$. The assertion to be proved now follows from a Corollary to the Inverse Function Theorem. □
Proposition. Suppose \((U, \psi)\) is a smooth local coordinate for \(M\). then \((\psi[U], \psi^{-1})\) is a smooth local parameter for \(M\).

**Proof.** We have only to verify that \(\psi[U]\) is open relative to \(M\) and this follows directly from the preceding Proposition. 

**Corollary.** Suppose \((T_i, \phi_i), i = 1, 2\), are smooth local parameters for \(M\)

\[ (T_1 \cap \phi_1^{-1} \circ \phi_2[T_2], \phi_2^{-1} \circ \phi_1, T_2 \cap \phi_2^{-1} \circ \phi_1[T_1]) \in \text{Diffeo}_m. \]

**Proof.** It is evident that the \(T_1 \cap \phi_1^{-1}[T_2]\) and \(T_2 \cap \phi_2^{-1}[T_2]\) are open sets of \(\mathbb{R}^m\) which are the domain and range of \(\phi_2^{-1} \circ \phi_1\). It is evident that \(\phi_2^{-1} \circ \phi_1\) is a univalent function.

Suppose \(t_1 \in T_1 \cap \phi_1^{-1} \circ \phi_2[T_2]\). Let \((U, \Phi, \mathbb{U}^n)\) be such that \(U\) is an open subset of \(V\), \(\phi_1(t_1) \in U\), \((U, \Phi, \mathbb{U}^n) \in \text{Diffeo}_n\), \(\Phi(\phi_1(t_1)) = 0\) and \(\Phi[M \cap U] = \mathbb{U}^{m,n}\). We have shown above that

\[ p_{m,n} \circ \Phi \circ \phi_i \in \text{Diffeo}_m, \ i = 1, 2. \]

Since

\[ (\phi_2^{-1} \circ \phi_i)[\phi_1^{-1}[U \cap \phi_2[T_2]]] = (p_{m,n} \circ \Phi \circ \phi_2)^{-1} \circ (p_{m,n} \circ \Phi \circ \phi_1) \]

and since \(\phi_1^{-1}[U \cap \phi_2[T_2]]\) is an open subset of \(\mathbb{R}^m\) containing \(t_1\) the proof is complete. 

**Lemma.** Suppose \((T_i, \phi_i), i = 1, 2\), are local parameters for \(M\), \(f \in \mathcal{L}_M^+\) and

\[ f(a) = 0 \quad \text{whenever} \ a \in \phi_1[T_1] \cap \phi_1[T_2]. \]

Then

\[ \int_{T_1} f \circ \phi_1 J_m \phi_1 = \int_{T_2} f \circ \phi_2 J_m \phi_2. \]

**Proof.** Replacing \(T_1\) with \(\phi_1^{-1}[T_2]\) and \(T_2\) by \(\phi_2^{-1}[T_1]\) we may assume that \(\phi_1[T_1] = \phi_2[T_2]\). From the preceding work we have that \((T_1, \phi_2^{-1} \circ \phi_1, T_2) \in \text{Diffeo}_m\). From the Change of Variables Formula for Multiple Integals we infer that

\[ \int_{T_2} f \circ \phi_2 J_m \phi_2 = \int_{T_1} f \circ \phi_2 (\phi_2^{-1} \circ \phi_1) J_m \phi_2 \circ (\phi_2^{-1} \circ \phi_1) |\det \partial (\phi_2^{-1} \circ \phi_1)|. \]

The Lemma will follow if we can show that

\[ J_m \phi_1 = (J_m \phi_2) \circ (\phi_2^{-1} \circ \phi_1) |\det \partial (\phi_2^{-1} \circ \phi_1)|. \]

Suppose \(t_1 \in T_1\) and let \(t_2 = \phi_2^{-1} \circ \phi_1(t_1) \in T_2\). By the Chain Rule we have

\[ \partial \phi_1(t_1) = \partial \phi_2(t_2) \circ \partial (\phi_2^{-1} \circ \phi_1)(t_1) \]

so that

\[ \partial \phi_1(t_1)^* = \partial (\phi_2^{-1} \circ \phi_1)(t_1)^* \circ \partial \phi_2(t_2)^*. \]

Thus

\[ \partial \phi_1(t_1)^* \circ \partial \phi_1(t_1) \]

\[ = \left( (\partial (\phi_2^{-1} \circ \phi_1)(t_1)^* \circ \partial \phi_2(t_2)^*) \circ (\partial \phi_2(t_2) \circ \partial (\phi_2^{-1} \circ \phi_1)(t_1)) \right) \]

\[ = \partial (\phi_2^{-1} \circ \phi_1)(t_1)^* \circ \left( (\partial \phi_2(t_2)^* \circ \partial \phi_2(t_2)) \circ \partial (\phi_2^{-1} \circ \phi_1)(t_1) \right) \]

so that, by the Product Rule for Determinants and the fact that

\[ \det L^* = \det L \quad \text{whenever} \ L \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^m) \]

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we find that

\[ J_m \phi_1(t_1) = |\text{det} \partial (\phi_2^{-1} \circ \phi_1)(t_1)| J_m \phi_2(t_2). \]

\( \Box \)

**Lemma.** There is a countable family \( T \) of smooth local parameters for \( M \) such that

\[ M = \cup \{ T : (T, \phi) \in T \}. \]

**Proof.** Exercise for the reader. \( \Box \)

**Lemma.** Suppose \( f \in \mathcal{L}_M^+ \) and

1. \((S_\mu, \alpha), \mu = 1, 2, 3, \ldots, \) are smooth local parameters for \( M; \)
2. \( \cup_{\mu=1}^{\infty} \text{rng} \alpha_\mu = M; \)
3. \( A_1 = \text{rng} \alpha_1 \) and \( A_\mu = \text{rng} \alpha_\mu \sim \cup_{\gamma<\mu} \text{rng} \alpha_\gamma \) if \( \mu > 1; \)
4. \( I = \sum_{\mu=1}^{\infty} \int_{S_\mu} (f 1_{A_\mu}) \circ \alpha_\mu J_m \alpha_\mu; \)
5. \((T_\nu, \beta_\nu), \nu = 1, 2, 3, \ldots, \) are smooth local parameters for \( M; \)
6. \( \cup_{\nu=1}^{\infty} \text{rng} \beta_\nu = M; \)
7. \( B_1 = \text{rng} \beta_1 \) and \( B_\nu = \text{rng} \beta_\nu \sim \cup_{\delta<\nu} \text{rng} \beta_\delta; \)
8. \( J = \sum_{\nu=1}^{\infty} \int_{T_\nu} (f 1_{B_\nu}) \circ \beta_\nu J_m \beta_\nu. \)

Thus

\[ I = J. \]

**Proof.** We have

\[ 1_M = \sum_{\mu=1}^{\infty} 1_{A_\mu} \quad \text{and} \quad 1_N = \sum_{\nu=1}^{\infty} 1_{B_\nu}. \]

Thus

\[ I = \sum_{\mu=1}^{\infty} \int_{S_\mu} (f 1_{A_\mu} 1_{B_\nu}) J_m \alpha_\mu \quad \text{and} \quad J = \sum_{\nu=1}^{\infty} \int_{T_\nu} (f 1_{A_\mu} 1_{B_\nu}) \circ \beta_\nu J_m \beta_\nu. \]

(Why?) That \( I = J \) now follows from a previous Proposition. \( \Box \)

**Theorem.** There is one and only one function

\[ I_M^+ : \mathcal{L}_M^+ \to [0, \infty] \]

such that

1. \( I_M^+ \left( \sum_{\nu=0}^{\infty} f_\nu \right) = \sum_{\nu=0}^{\infty} I_M^+ (f_\nu) \) whenever \( f \) is a sequence in \( \mathcal{L}_M^+; \)
2. \( I_M^+ (f) = \int_T f \circ \phi J_m \phi \) whenever \( f \in M \) and \( f(a) = 0 \) whenever \( a \not\in \text{rng} \phi. \)
Proof. Combine the preceding two Lemmas. □

Definition. We let

\[ \mathcal{L}_M \]

be the vector space of Lebesgue measurable function \( f : M \to \mathbb{R} \) such that \( I_M^+(|f|) < \infty \). For each \( f \in \mathcal{L}_M \) we let

\[ I_M(f) = I_M^+(f^+) - I_M^-(f^-). \]