1. Integration theory, Part Two.

1.1. The Monotone Convergence Theorem. The following Theorem is fundamental.

Theorem 1.1. (The Monotone Convergence Theorem). Suppose $f$ is a non-decreasing sequence in $\mathcal{F}_n^+$. Then

$$I(\sup_{\nu} f_{\nu}) = \sup_{\nu} I(f_{\nu}).$$

Proof. Let $a$ and $b$ be the left and right hand sides of (1), respectively. Owing to the monotonicity of $I$, we find that $b \leq a$. Thus we need only show that $a \leq b$ and we may assume that $b < \infty$.

To this end, let $\epsilon > 0$. For each $\nu \in \mathbb{N}$ let $s_{\nu} \in S_{n,1}^+$ be such that $f_{\nu} \leq \sup s_{\nu}$ and

$$I_{n,1}^+(s_{\nu}) \leq I(f_{\nu}) + 2^{-\nu-1} \epsilon.$$

For each $\mu, \nu \in \mathbb{N}$ with $\mu \leq \nu$ we let

$$S_{\mu}^\nu = \bigvee_{n=\mu}^{\nu} s_n \in I_{n,1}^+.$$

We define the sequence $t$ by letting

$$t_{\nu} = (S_{\nu}^0)_{\nu} \in S_{n}^+.$$

For any $\nu \in \mathbb{N}$ we have

$$t_{\nu} = (S_{\nu}^0)_{\nu} \leq (S_{0}^{\nu+1})_{\nu} \leq (S_{0}^{\nu+1})_{\nu+1} = t_{\nu+1}$$

so $t \in S_{n,1}^+$ and

$$I_{n,1}^+(t_{\nu}) = I_{n,1}^+((S_{0}^\nu)_{\nu}) \leq I_{n,1}^+(S_{0}^\nu).$$

Moreover, for any $\nu, \xi \in \mathbb{N}$, we have

$$(s_{\nu})_{\nu} \leq (s_{\nu})_{\nu \vee \xi} \leq (S_{0}^{\nu} \vee \xi)_{\nu \vee \xi} = (S_{0}^{\nu} \vee \xi)_{\nu \vee \xi} \leq \sup t;$$

it follows that $f_{\nu} \leq \sup t$ for any $\nu \in \mathbb{N}$ which in turn implies that $\sup f \leq \sup t$ so

$$I(\sup f) \leq I_{n,1}^+(t).$$

We will complete the proof by showing that

$$I_{n,1}^+(t) \leq \sup_{\nu} I(f_{\nu}) + \epsilon.$$

Suppose $\mu, \nu \in \mathbb{N}$ and $\mu < \nu$. Since $s_{\mu} \leq S_{\mu}^\nu$ we have

$$f_{\mu} \leq f_{\nu} \wedge f_{\nu+1} \leq (\sup s_{\nu}) \wedge (\sup S_{\mu+1}^\nu) = \sup(s_{\nu} \wedge S_{\mu+1}^\nu).$$

Using the fact that $a \wedge b = a + b$ whenever $a, b \in [0, \infty]$ we find that

$$s_{\mu} \wedge S_{\mu+1}^\nu + S_{\mu}^\nu = s_{\mu} \wedge S_{\mu+1} + s_{\mu} \vee S_{\mu+1}^\nu = s_{\mu} + S_{\mu+1}^\nu.$$
thus
\[
I(f_\mu) + I_{n,1}^+(S_{\mu}^\nu) \leq I_{n,1}^+(s_\mu \land S_{\mu+1}^n) + I_{n,1}^+(S_{\mu}^\nu)
\]

\[
= I_{n,1}^+(s_\mu \land S_{\mu+1}^n + S_{\mu}^\nu)
\]

\[
= I_{n,1}^+(s_\mu + S_{\mu+1}^n)
\]

\[
= I_{n,1}^+(s_\mu) + I_{n,1}^+(S_{\mu+1}^n)
\]

\[
\leq I(f_\mu) + 2^{-\mu-1} \epsilon + I_{n,1}^+(S_{\mu+1}^n).
\]

Since \(I(f_\mu) < \infty\) we obtain
\[
I_{n,1}^+(S_{\mu}^\nu) \leq I_{n,1}^+(S_{\mu+1}^n) + 2^{-\mu-1} \epsilon; 
\]

Summing from \(\mu = 0\) to \(\nu\) and using (1) we find that
\[
I_{n}^+(t_\nu) \leq I_{n,1}^+(S_{\nu}^n) \leq I_{n,1}^+(S_{\nu}^\nu) + \epsilon = I_{n,1}^+(s_\nu) + \epsilon < I(f_\nu) + \epsilon^{-\nu-1} + \epsilon 
\]

thereby establishing (2).

\[\square\]

**Corollary 1.1. (Fatou’s Lemma.)** Suppose \(f\) is a sequence in \(\mathcal{F}_n^+\). Then
\[
I(\liminf_{\nu} f_\nu) \leq \liminf_{\nu} I(f_\nu).
\]

**Proof.** Let \(F_\nu = \inf_{0 \leq \mu \leq \nu} f_\mu\) for each \(\nu \in \mathbb{N}\) and apply the Monotone Convergence Theorem to \(F\).

\[\square\]

1.2. Basic theory of Lebesgue integration.

**Theorem 1.2.** Suppose \(f \in \mathcal{F}_n^+ \cap \text{Leb}_n\). Then
\[
I(f) = L(f).
\]

**Proof.** Let \(\epsilon > 0\). By a previous Proposition there is \(s \in \mathcal{S}_n^+\) such that \(I(f-s) < \epsilon/2\).
Since \(I(s) = I_n(s) = L(s)\) it follows that
\[
|I(f) - L(f)| \leq |I(f) - I(s)| + |L(f-s)| \leq 2I(|f-s|) \leq \epsilon.
\]

Thus (ii) holds.

\[\square\]

**Lemma 1.1.** Suppose \(f\) is a sequence in \(\mathcal{F}_n^+ \cap \text{Leb}_n\) such that

(i) \(\sup_{\nu} f_\nu(x) < \infty\) for each \(x \in \mathbb{R}^n\) and

(ii) \(I(\sup_{\nu} f_\nu) \leq \infty\).

Then \(\sup_{\nu} f_\nu \in \text{Leb}_n\).

**Proof.** Replacing \(f_\nu\) by \(\sup_{0 \leq \mu \leq \nu} f_\mu\) if necessary we may assume without loss of generality that \(f\) is nondecreasing.

Let \(\epsilon > 0\). Since (ii) holds we may choose \(N \in \mathbb{N}\) such that
\[
\sup_{\nu} I(f_\nu) \leq I(f_N) + \epsilon.
\]

It follows from the preceding Proposition that
\[
I(f_\nu - f_N) = L(f_\nu - f_N) = L(f_\nu) - L(f_N) = I(f_\nu) - I(f_N)
\]

for any \(\nu \in \mathbb{N}\) so that
\[
\sup_{\nu} I(f_\nu - f_N) \leq \epsilon.
\]
Since $f$ is nondecreasing we may use the Monotone Convergence Theorem to infer that
\[ l(\sup_{\nu} f_{\nu}) - f_N = l(\sup_{\nu} (f_{\nu} - f_N)) = \sup_{\nu} l(f_{\nu} - f_N) \leq \epsilon. \]
\[ \square \]

**Lemma 1.2.** Suppose $f$ is a sequence in $\mathcal{F}_n^+ \cap \text{Leb}_n$. Then $\inf_{\nu} f_{\nu} \in \text{Leb}_n$.

**Proof.** For each $\nu \in \mathbb{N}$ let $F_{\nu} = \inf_{0 \leq \mu \leq \nu} f_{\mu} \in \text{Leb}_n$. Evidently, $F$ is nonincreasing so $\mathbb{N} \ni \nu \mapsto F_0 - F_{\nu}$ is nondecreasing. Since
\[ \inf_{\nu} F_{\nu} = F_0 - \sup_{\nu} (F_0 - F_{\nu}) \]
and since $\inf_{\nu} f_{\nu} = \inf_{\nu} F_{\nu}$ this Lemma follows from Lemma 1.2. \[ \square \]

**Theorem 1.3.** Suppose $F \in \mathcal{F}_n^+ \cap \text{Leb}_n$ and $l(F) < \infty$ and there is a sequence $f$ in $\text{Leb}_n$ such that
\[ F(x) = \lim_{\nu \to \infty} f_{\nu}(x) \quad \text{for} \quad x \in \mathbb{R}^n. \]
Then $F \in \text{Leb}_n$.

**Proof.** Choose a $s \in \mathcal{S}_{n,1}^+$ such that $F \leq \sup s$ and $I_{n,1}^+(s) < \infty$. Using the Lemmas 1.1 and 1.2 we infer that, for each $\xi \in \mathbb{N}$,
\[ F \land s_\xi = \inf_{\nu \geq \nu} f_{\mu} \land s_\xi \in \text{Leb}_n. \]
Since $F = \sup_\xi F \land s_\xi$ the Theorem follows from the Lemma 1.1. \[ \square \]

**Theorem 1.4.** (The Lebesgue Dominated Convergence Theorem.) Suppose
\begin{enumerate}
  \item $f$ is a sequence in $\text{Leb}_n$ and $F \in \mathcal{F}_n$ is such that
  \[ \lim_{\nu \to \infty} f_{\nu}(x) = F(x) \quad \text{for all} \quad x \in \mathbb{R}^n; \]
  \item $g$ is a sequence in $\text{Leb}_n$ such that
  \[ |f_{\nu}| \leq g_{\nu}, \quad \nu \in \mathbb{N}; \]
  \item $G \in \mathcal{F}_n$,
  \[ \lim_{\nu \to \infty} g_{\nu}(x) = G(x) \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad \lim_{\nu \to \infty} l(g_{\nu}) = l(G) < \infty. \]
\end{enumerate}
Then $F \in \text{Leb}_n$ and
\[ \lim_{\nu \to \infty} l(|F - f_{\nu}|) = 0. \]

**Proof.** For each $\nu \in \mathbb{N}$ let $h_{\nu} = G + g_{\nu} - |F - f_{\nu}| \in \mathcal{F}_n^+ \cap \text{Leb}_n$. We know from the previous Theorem that $G$ and $|F - f_{\nu}| = \lim_{\mu \to \infty} |f_{\mu} - f_{\nu}|$, $\nu \in \mathbb{N}$ are in $\text{Leb}_n$. Thus, for any $\nu \in \mathbb{N}$,
\[ L(h_{\nu}) = L(G) + L(g_{\nu}) - L(|F - f_{\nu}|) \]
so
\[ l(h_{\nu}) = l(G) + l(g_{\nu}) - l(|F - f_{\nu}|). \]
By Fatou’s Lemma we have
\[ 2l(G) = l(\lim_{\nu \to \infty} h_{\nu}) \leq \lim inf_{\nu \to \infty} l(h_{\nu}). \]
Since
\[ \liminf_{\nu \to \infty} l(h_{\nu}) = 2l(G) - \limsup_{\nu \to \infty} l(|F - f_{\nu}|) \]
it follows that
\[ \limsup_{\nu \to \infty} l(|F - f_{\nu}|) = 0. \]
This in turn implies that \( F \in \text{Leb}_n. \) \( \square \)

1.3. Lebesgue measurable sets.

**Theorem 1.5.** Suppose \( A \) in a nondecreasing sequence of subsets of \( \mathbb{R}^n \). Then
\[ |\bigcup_{\nu=0}^{\infty} A_{\nu}|^* = \sup_{\nu} |A_{\nu}|^*. \]

*Proof.* Since \( l^+(1_B) = |B|^* \) for any subset \( B \) of \( \mathbb{R}^n \) this follows from the Monotone Convergence Theorem. \( \square \)

**Theorem 1.6.** Suppose \( E \subset \mathbb{R}^n \). The following are equivalent:
(i) \( 1_E \in \text{Leb}_n \);
(ii) for each \( \epsilon > 0 \) there is a multirectangle \( M \) such that
\[ |(E \sim M) \cup (M \sim E)|^* \leq \epsilon. \]

*Proof.* Suppose (i) holds and \( \epsilon > 0 \). Choose \( s \in \mathcal{S}_n^+ \) such that \( l(|1_E - s|) \leq \epsilon / 2 \), let \( M = \{ x \in \mathbb{R}^n : s(x) \geq 1/2 \} \) and note that \( M \) is a multirectangle. Since
\[ \frac{1}{2} l((E \sim M) \cup (M \sim E)) = \frac{1}{2} |1_E - 1_M| \leq |1_E - s| \]
we find that \( |(E \sim M) \cup (M \sim E)|^* \leq \epsilon. \) Thus (ii) holds.

Suppose (ii) holds and \( \epsilon > 0 \). Then there is a multirectangle \( M \) such that \( l(|1_E - 1_M|) \leq \epsilon. \) Thus \( 1_E \in \text{Leb}_n \) since \( 1_M \in \mathcal{S}_n. \) \( \square \)

**Definition 1.1.** We let
\[ \text{Leb}_n^+ \]
be the family of functions in \( \mathcal{F}_n^+ \) which are the supremum of a nondecreasing sequence \( \mathcal{F}_n^+ \cap \text{Leb}_n. \)

**Theorem 1.7.** Suppose \( F, G \in \text{Leb}_n^+ \). Then
\[ 1^+(F + G) = 1^+(F) + 1^+(G). \]

*Proof.* Let \( f, g \) be nondecreasing sequences \( \mathcal{F}_n^+ \cap \text{Leb}_n \) with suprema \( F \) and \( G \), respectively. Using the Monotone Convergence Theorem three times we calculate
\[ 1^+(F + G) = \sup_{\nu} 1^+(f_{\nu} + g_{\nu}) \]
\[ = \sup_{\nu} L(f_{\nu} + g_{\nu}) \]
\[ = \sup_{\nu} L(f_{\nu}) + L(g_{\nu}) \]
\[ = \sup_{\nu} 1^+(f_{\nu}) + 1^+(g_{\nu}) \]
\[ = 1^+(f) + 1^+(g). \] \( \square \)
Definition 1.2. We say a subset $E$ of $\mathbb{R}^n$ is Lebesgue measurable if $1_E \in \text{Leb}_n^+$ in which case we let

$$\mathcal{L}^n(E) = 1^+(1_E)$$

and call this nonnegative extended real number the Lebesgue measure of $E$.

Theorem 1.8. Suppose $E \subset \mathbb{R}^n$. Then $E$ is Lebesgue measurable if for each $\epsilon > 0$ there is a countable family $\mathcal{R}$ of rectangles such that

$$|(E \sim (\cup \mathcal{R}) \cup ((\cup \mathcal{R}) \sim E))|^* \leq \epsilon.$$ 

Theorem 1.9. The following statements hold.

(i) $\mathcal{M}_n \subset \mathcal{L}_n$.

(ii) $E \in \mathcal{L}_n$ and $|E|^* < \infty$ if and only if for each $\epsilon > 0$ there is a multirectangle $M$ such that

$$|(E \sim M) \cup (M \sim E)|^* \leq \epsilon.$$ 

(iii) $E \in \mathcal{L}_n$ if and only if there is a nondecreasing sequence $F$ in $\{G \in \mathcal{L}_n : |G|^* < \infty\}$ such that $E = \cup_{\nu=0}^\infty F_{\nu}$.

(iv) If $E, F \in \mathcal{L}_n$ then $E \cup F, E \cap F, E \sim F \in \mathcal{L}_n$ and

$$|E \cup F|^* + |E \cap F|^* = |E|^* + |F|^*.$$ 

If $\mathcal{E}$ is a countable nonempty family of Lebesgue measurable subsets of $\mathbb{R}^n$ the following assertions hold:

(v) $\cup \mathcal{E}$ and $\cap \mathcal{E}$ are Lebesgue measurable;

(vi) if $\mathcal{E}$ is disjointed then

$$|\cup \mathcal{E}|^* = \sum_{E \in \mathcal{E}} |E|^*;$$ 

(vii) if $\mathcal{E}$ is nested then

$$|\cup \mathcal{E}|^* = \sup\{|E|^* : E \in \mathcal{E}\};$$ 

(viii) if $\mathcal{E}$ is nested and $|E|^* < \infty$ for some $E \in \mathcal{E}$ then

$$|\cap \mathcal{E}|^* = \inf\{|E|^* : E \in \mathcal{E}\}.$$

Proof. Exercise for the reader. $\square$

Remark 1.1. In particular, the Lebesgue measurable subsets of $\mathbb{R}^n$ form a $\sigma$-algebra of subsets of $\mathbb{R}^n$.

Definition 1.3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$. We say $f$ is Lebesgue measurable if $f^{-1}[U] \in \mathcal{L}_n$ whenever $U$ is an open subset $\mathbb{R}^n$.

Proposition 1.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$. The following are equivalent.

(i) $f$ is Lebesgue measurable.

(ii) $\{x \in \mathbb{R}^n : f(x) > c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$.

(iii) $\{x \in \mathbb{R}^n : f(x) \geq c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$.

(iv) $\{x \in \mathbb{R}^n : f(x) < c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$.

(v) $\{x \in \mathbb{R}^n : f(x) \leq c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$. 

Proof. Since

\[ \{ x \in \mathbb{R}^n : f(x) \geq c \} - \cap_{\nu} = 1^\infty \{ x \in \mathbb{R}^n : f(x) > c - \frac{1}{n} \} \]

we see that that (ii) implies (iii). Since

\[ \{ x \in \mathbb{R}^n : f(x) < c \} = \mathbb{R}^n \sim \{ x \in \mathbb{R}^n : f(x) \geq c \} \]

we see that (iii) implies (iv). Since

\[ \{ x \in \mathbb{R}^n : f(x) \leq c \} = \cap_{\nu=1}^\infty \{ x \in \mathbb{R}^n : f(x) < c + \frac{1}{\nu} \} \]

we see that (iv) implies (v). Since

\[ \{ x \in \mathbb{R}^n : f(x) > c \} = \mathbb{R}^n \sim \{ x \in \mathbb{R}^n : f(x) \leq c \} \]

we see that (v) implies (ii). Thus (ii),(iii),(iv) and (v) are equivalent. Since

(i) obviously implies (ii). Suppose (ii) holds. Then, as (iv) holds,

\[ \{ x \in \mathbb{R}^n : a < f(x) < b \} \in \mathcal{L}_n \]

whenever \(-\infty < a < b < \infty\).

Let \( U \) be an open subset of \( \mathbb{R} \). Let \( \mathcal{I} \) be the family of open subintervals of \( U \) with rational endpoints. Then, as \( \mathcal{I} \) is countable, we find that

\[ f^{-1}[U] = \cup \{ f^{-1}[I] : I \in \mathcal{I} \} \in \mathcal{L}_n \]

. Thus (i) holds. \( \square \)

Corollary 1.2. Suppose \( N \) is a positive integer, \( f_i : \mathbb{R}^n \to \mathbb{R} \), \( i = 1, \ldots, N \) are Lebesgue measurable functions, and

\[ M : \mathbb{R}^N \to \mathbb{R} \]

is continuous. Then

\[ \mathbb{R}^n \ni x \mapsto M(f_1(x), \ldots, f_N(x)) \]

is Lebesgue measurable.

Corollary 1.3. The set of Lebesgue measurable functions is closed under the arithmetic operation as well as the lattice operations.

Proposition 1.2. Suppose \( f \) is a sequence of Lebesgue measurable functions and \( F : \mathbb{R}^n \to \mathbb{R} \) is such that

\[ \lim_{\nu \to \infty} f_\nu(x) = F(x) \]

whenever \( x \in \mathbb{R}^n \).

Then \( F \) is Lebesgue measurable.

Proof. Suppose \( c \in \mathbb{R} \). Then

\[ \{ x \in \mathbb{R}^n : F(x) > c \} = \cup_{n=1}^\infty \cup_{N=0}^\infty \cap_{\nu=N}^\infty \{ x \in \mathbb{R}^n : f_\nu(x) > c + \frac{1}{n} \} \]

\( \square \)

Lemma 1.3. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \), \( c \in \mathbb{R} \), \( E = \{ x \in \mathbb{R}^n : f(x) > c \} \) and

\[ g_h(x) = \frac{1}{h} [f \land (c + h) - f \land c] \]

for \( h \in (0, \infty) \).

Then

(i) \( g_h \leq g_k \) if \( 0 < k < h < \infty \);

(ii) \( 1_E = \sup_{0<h<\infty} g_h \).
Proof. To prove (i) we suppose \( a \in \mathbb{R}^n \) and \( 0 < k < h < \infty \) and we observe that
\[
\begin{align*}
f(a) < c & \Rightarrow g_k(a) = 0 = g_{k}(a), \\
c \leq f(a) < c + k & \Rightarrow g_k(a) = \frac{1}{k} [f(a) - c] \leq \frac{1}{k} [f(a) - c] = g_{k}(a), \\
c + k \leq f(a) < c + h & \Rightarrow g_k(a) = \frac{1}{h} [f(a) - c] \leq 1 = g_{k}(a), \\
c + h \leq f(a) & \Rightarrow g_k(a) = 1 = g_{k}(a).
\end{align*}
\]

(ii) is evident. \( \square \)

**Lemma 1.4.** Suppose \( f : \mathbb{R}^n \to \mathbb{R} \); \( c \) is a a sequence of positive real numbers such that
\[
\lim_{\nu \to \infty} c_{\nu} = 0 \quad \text{and} \quad \sum_{\nu=0}^{\infty} c_{\nu} = \infty;
\]
and \( E \) is the sequence of subsets of \( \mathbb{R}^n \) defined inductively by setting \( E_0 = \{ x \in \mathbb{R}^n : f(x) > c_0 \} \) and requiring that
\[
E_{\nu+1} = \{ x \in \mathbb{R}^n : f(x) > \sum_{\mu=0}^{\nu} c_{\mu} \mathbf{1}_{E_{\mu}} \} \quad \text{whenever} \ \nu > 0.
\]
Then
\[
f = \sum_{\nu=0}^{\infty} c_{\nu} \mathbf{1}_{E_{\nu}}.
\]

**Proof.** Straightforward exercise. \( \square \)

**Theorem 1.10.** Suppose \( f : \mathbb{R}^n \to \mathbb{R} \). Then \( f \in \text{Leb}_n \) if and only if \( \text{I}(f) < \infty \) and \( f \) is Lebesgue measurable.

**Proof.** Suppose \( f \in \text{Leb}_n \). Let \( c \in \mathbb{R} \). That \( \{ x \in \mathbb{R}^n : f(x) > c \} \in \mathcal{L}_n \) follows the first of the two preceding Lemmas and our earlier theory.

Suppose \( \text{I}(f) < \infty \) and \( f \) is Lebesgue measurable. Writing \( f = f^+ - f^- \) we see we need only consider the case \( f \geq 0 \). Let \( c \) be a sequence of positive real numbers such that \( \lim_{\nu \to \infty} c_{\nu} = 0 \) and \( \sum_{\nu=0}^{\infty} c_{\nu} = \infty \) and let the sequence \( E \) be as in the preceding Lemma so that
\[
f = \sum_{\nu=0}^{\infty} c_{\nu} \mathbf{1}_{E_{\nu}}.
\]
Note that \( E_{\nu} \in \mathcal{L}_n \). That \( f \in \text{Leb}_n \) follows from earlier theory. \( \square \)

**Theorem 1.11. (The absolute continuity of the integral.)** Suppose \( f \in \text{Leb}_n \). Then for each \( \epsilon > 0 \) there is \( \delta > 0 \) such that
\[
E \in \mathcal{L}_n \text{ and } |E| < \delta \Rightarrow \text{L}_E(|f|) < \epsilon.
\]

**Proof.** For each nonnegative integer \( \nu \) let \( g_{\nu} = |f| \wedge \nu \). Since \( g_{\nu} \uparrow |f| \) as \( \nu \uparrow \infty \) we infer from the Monotone Convergence Theorem that \( \text{I}(g_{\nu}) \uparrow \text{I}(|f|) \) as \( \nu \uparrow \infty \). Choose a positive integer \( N \) such that
\[
\text{I}(|f|) - \text{I}(g_N) < \frac{\epsilon}{2}.
\]
By the preceding theory, \( g_N \in \text{Leb}_n \). Let \( \delta = \frac{\epsilon}{2N} \). If \( E \in \mathcal{L}_n \) and \( |E| < \delta \) then

\[
|f|_{1E} = (|f| - g_N)_{1E} + g_N 1_E \leq |f| - g_N + N 1_E
\]

so that

\[
L_E(|f|) = L(|f|_{1E}) \leq L(|f| - g_N + N 1_E) = L(|f|) - L(g_N) + N|E| < \epsilon.
\]

\[\square\]

**Theorem 1.12. (Minkowski’s inequality in integral form.)** Suppose \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is Lebesgue measurable and \( 1 \leq p < \infty \). Then

\[
\left( \int \left( \int |h(x, y)|^p \, dx \right)^{1/p} \, dy \right)^{1/p} \leq \int \left( \int |h(x, y)|^p \, dx \right)^{1/p} \, dy.
\]

**Remark 1.2.** We haven’t shown that either of the above integrals is defined because we haven’t done product integration.

**Proof.** By an approximation argument we need only consider \( h \) of the form

\[
h(x, y) = \sum_{j=1}^{N} f_j(x) 1_{F_j}(y), \quad (x, y) \in \mathbb{R} \times \mathbb{R},
\]

where \( N \) is a positive integer; \( f_j \) is Lebesgue measurable and \( F_j \in \mathcal{L}_n, j = 1, \ldots, N; \) and \( F_i \cap F_j = \emptyset \) if \( 1 \leq i < j \leq N \). We use Minkowski’s inequality to estimate

\[
\left( \int \left( \int h(x, y) \, dy \right)^{1/p} \, dx \right)^{1/p} = \left( \int \left( \sum_{j=1}^{N} ||f_j||_{L^p}^p \, dx \right)^{1/p} \right)^{1/p} \leq \sum_{j=1}^{N} ||F_j|| \left( \int |f_j(x)|^p \, dx \right)^{1/p}.
\]

But

\[
\int \left( \int |h(x, y)|^p \, dx \right)^{1/p} \, dy = \sum_{j=1}^{N} \int_{F_j} \left( \int |h(x, y)|^p \, dx \right)^{1/p} = \sum_{j=1}^{N} \int_{F_j} \left( \int |f_j(x)|^p \, dx \right)^{1/p}.
\]

\[\square\]

**Corollary 1.4.** Suppose \( f \) and \( g \) are Lebesgue measurable. Then

\[
||f \ast g||_p \leq ||f||_p ||g||_1.
\]

**Proof.** Using Minkowski’s Inequality in integral form we estimate

\[
||f \ast g||_p = \left( \int \left( \int |f(x) - y)g(y) \, dy \right)^{p} \, dx \right)^{1/p} \leq \left( \int \left( \int |f(x) - y)g(y) \, dx \right)^{p} \, dy \right)^{1/p} = \left( \int \left( \int |f(x) \, dx \right)^{p} \, dy \right)^{1/p} ||g||_1.
\]

\[\square\]
Proposition 1.3. Suppose $1 \leq p < \infty$, $f$ is Lebesgue measurable and
\[
\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p} < \infty.
\]
Then for each $\epsilon > 0$ there is an elementary function $s$ such that $\|f - s\|_p < \epsilon$.

Proof. Let $\epsilon > 0$.

For each positive integer $\nu$ let $E_\nu = \{x \in \mathbb{R}^n : |f(x)| \leq \nu\}$. Since $1_{E_\nu} |f|^p \uparrow |f|^p$ as $\nu \uparrow \infty$ we infer from the Monotone Convergence Theorem and the additivity of the integral that
\[
\int_{E_\nu} |f(x)|^p \, dx \uparrow \int |f(x)|^p \, dx \quad \text{as} \quad \nu \uparrow \infty.
\]
By the additivity of the integral we infer that
\[
\|f - 1_{E_\nu} f\|_p^p = \int_{\mathbb{R}^n \sim E_\nu} |f(x)|^p \, dx = \int |f(x)|^p \, dx - \int_{E_\nu} |f(x)|^p \, dx \downarrow 0 \quad \text{as} \quad \nu \uparrow \infty.
\]
We may therefore choose a positive integer $N$ such that $\|f - 1_{E_N} f\|_p \leq \epsilon/2$. Since $f 1_{E_N} \in \text{Leb}_1$ we may choose an elementary function $s$ such that $|s| \leq M$ and
\[
(2M)^p \int |f 1_{E_N} - s|(x) \, dx \leq \left(\frac{\epsilon}{2}\right)^p.
\]
Then
\[
\|f 1_{E_N} - s\|_p^p = \int |f 1_{E_N} - s|^p \, dx \leq (2M)^p \int |f 1_{E_N} - s| \, dx \leq \left(\frac{\epsilon}{2}\right)^p.
\]
It follows from Minkowski's Inequality that
\[
\|f - s\|_p \leq \|f - 1_{E_N} f\|_p + \|1_{E_N} f - s\|_p \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

1.3.1. Smoothing. Let
\[
\phi : \mathbb{R}^n \rightarrow \mathbb{R}
\]
be a smooth function such that
\begin{enumerate}
  \item $0 \leq \phi$;
  \item $\{x \in \mathbb{R}^n : \phi(x) \neq 0\} \subset \{x \in \mathbb{R}^n : |x| < 1\}$;
  \item $\int \phi(x) \, dx = 1$.
\end{enumerate}
For each $\epsilon > 0$ we let
\[
\phi_\epsilon(x) = \frac{1}{\epsilon^n} \phi\left(\frac{x}{\epsilon}\right) \quad \text{for} \quad x \in \mathbb{R}^n.
\]
Then
\begin{enumerate}
  \item $0 \leq \phi_\epsilon$;
  \item $\{x \in \mathbb{R}^n : \phi_\epsilon(x) \neq 0\} \subset \{x \in \mathbb{R}^n : |x| < \epsilon\}$;
  \item $\int \phi_\epsilon(x) \, dx = 1$.
\end{enumerate}

Theorem 1.13. Suppose $1 \leq p < \infty$ and $f$ is measurable and
\[
\int |f(x)|^p \, dx < \infty.
\]
Then $\phi_\epsilon * f$ is smooth and
\[
\|f - \phi_\epsilon * f\|_p \rightarrow 0 \quad \text{as} \quad \epsilon \downarrow 0.
\]
Proof. Let $\eta > 0$ and let $s$ be a elementary function such that $\|f - s\|_p < \eta/3$. Then

$$\|f - \phi * f\|_p \leq \|f - s\|_p + \|s - \phi * s\|_p + \|\phi * (f - s)\|_p 2\eta + \|s - \phi * s\|_p.$$ 

Finally, as $s$ is elementary, $\|s - \phi * s\|_p \rightarrow 0$ as $\epsilon \downarrow 0$. (Do you see why?) $\square$