The Implicit Function Theorem. Suppose

(1) X, Y and Z are Banach spaces;

(2) C is an open subset of $X \times Y$,

$$f : C \to Z$$

and f is continuously differentiable on C;

(3) $(a, b) \in C$ and

$$Y \ni v \mapsto \partial f(a, b)(0, v)$$

is a Banach space isomorphism from Y onto Z;

Then there are an open subset U of X such that $a \in U$; an open subset W of Z such that $f(a, b) \in W$; an open subset V of $X \times Y$ such that $(a, b) \in V \subset C$; and g such that

(4) $g : U \times W \to Y$ and g is continuously differentiable on $U \times W$;

(5)

$$(x, y) \in V \text{ and } z = f(x, y) \quad \Leftrightarrow \quad (x, z) \in U \times W \text{ and } y = g(x, z).$$

0.1. Remark. Note that $C = \{(x, g(x, z)) : (x, z) \in U \times W\}$.

Proof. Let $F(x, y) = (x, f(x, y))$ for $x \in C$. By the Corollary to the Inverse Function Theorem, the Chain Rule and the smoothness of inversion we obtain an open subset D of X such that $(a, b) \in D \subset C$ and

(6) $F[D]$ is an open subset of $Y \times Z$;

(7) $F|D$ is univalent;

(8) $(F|D)^{-1}$ is continously differentiable.

Let U and W be open subsets of Y and Z, respectively, such that $a \in U$, $c \in W$ and $U \times W \subset F[D]$. Let $G = (F|D)^{-1}$ and let $V = G[U \times W]$. Let $g : U \times W \to Y$ and $i : U \times V \to X$ be such that $G(x, z) = (i(x, z), g(x, z))$ whenever $(x, z) \in U \times W$. Since

$$(x, z) = F(G(x, z)) = (i(x, z), f(x, g(x, z))) \quad \text{whenever } (x, z) \in U \times W$$

we find that

$$i(x, z) = x \quad \text{whenever } (x, z) \in U \times W.$$

We have only to let $V = G[U \times W]$. □

The Theorem on Functional Dependence. Suppose

(1) m and n are positive integers

(2) C is an open subset of $\mathbf{R}^m \times \mathbf{R}^n$,

$$f : C \to \mathbf{R}^n$$

and f is continuously differentiable;

(3) $(a, b) \in C$ and

$$\mathbf{R}^n \ni v \mapsto \partial f(a, b)(0, v)$$

carries $\mathbf{R}^n$ isomorphically onto itself;

(4) $\varphi : C \to \mathbf{R}$, φ is continuously differentiable and

$$\partial\varphi(x, y) \in \mathbf{span}\,\{\partial f^i(x, y); i = 1, \ldots, n\}$$

whenever $(x, y) \in C$.

Then there are an open subset W of $\mathbf{R}^n$ such that $f(a, b) \in V$, an open subset V of $\mathbf{R}^m \times \mathbf{R}^n$ such that $(a, b) \in V \subset C$ and Φ such that

$$\Phi : W \to \mathbf{R},$$

$f(x, y) \in W$ if $(x, y) \in V$ and

$$\varphi(x, y) = \Phi(f(x, y)) \quad \text{whenever } (x, y) \in V. \tag{5}$$

Proof. We use the previous Theorem to obtain an open subset U of $\mathbf{R}^m$ such that $a \in U$; an open subset W of $\mathbf{R}^n$ such that $f(a, b) \in W$; an open subset V of $\mathbf{R}^m \times \mathbf{R}^n$ such that $(a, b) \in V \subset C$; and g such that

(5) $g : U \times W \to V$ and g is continuously differentiable;

$$(x, y) \in V \text{ and } z = f(x, y) \quad \Leftrightarrow \quad (x, z) \in U \times W \text{ and } y = g(x, z). \tag{6}$$

We may assume that U is connected.

It follows from (4) that there is a unique function c on V with values in the dual space of $\mathbf{R}^n$ such that

$$\partial\varphi(x, y) = c(x, y) \circ \partial f(x, y) \quad \text{whenever } (x, y) \in V.$$

Let $G(x, z) = (x, g(x, z))$ and let $q(x, z) = z$ for $(x, z) \in U \times W$. From the Chain Rule we obtain

$$\begin{aligned}
\partial(\varphi \circ G)(x, z) &= \partial\varphi(G(x, z)) \circ \partial G(x, z) \\
&= c(G(x, z)) \circ \partial f(G(x, z)) \circ \partial G(x, z) \\
&= c(G(x, z)) \circ \partial(f \circ G)(x, z) \\
&= c(G(x, z)) \circ q
\end{aligned}$$

so that

$$\partial(\varphi \circ G)(x, z)(u, 0) = 0 \quad \text{whenever } u \in \mathbf{R}^m$$

whenever $(x, z) \in U \times W$. Thus, as U is connected, we infer that

$$\varphi(x, g(x, z)) = \varphi(a, g(a, z)) \quad \text{whenever } (x, z) \in U \times W.$$

Let

$$\Phi(z) = \varphi(a, g(a, z)) \quad \text{for } z \in W.$$

Evidently, $\varphi \circ G(x,z) = \Phi(f(G(x,z)))$ for $(x,z) \in U \times W$ from which we infer that (5) holds. $\square$