1. **Homework Eight. Due Friday, October 23.**

1.1. **An exercise on difference quotients.** Suppose $I$ is an open interval, $a \in I$, $f : I \to \mathbb{R}$ and $f$ is differentiable at $a$.

Show that for each $\epsilon > 0$ there is $\delta > 0$ such that
\[
|a - \delta < x < a \text{ and } a < y < a + \delta \Rightarrow \left| \frac{f(y) - f(x)}{y - x} - f'(a) \right| < \epsilon.
\]

Show by example that it is not necessarily the case that $a < x < a + \delta$ and $a < y < a + \delta \Rightarrow |f(y) - f(x) - f'(a)(y - x)| < \epsilon$.

1.2. **An exercise on differentiation.** Suppose $I$ is an open interval, $a \in I$, $f : I \to \mathbb{R}$, $f$ is differentiable at each point of $I \sim \{a\}$, $f$ is continuous at $a$ and
\[
\lim_{x \to a} f'(x) = L
\]
for some $L \in \mathbb{R}$. Prove that $f$ is differentiable at $a$ and $f'(a) = L$.

1.3. **A very useful example.** We define
\[
\phi : \mathbb{R} \to \mathbb{R}
\]
by requiring that
\[
\phi(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
e^{-\frac{1}{x}} & \text{if } x > 0.
\end{cases}
\]

Show that $\text{dmn } \phi^{(m)} = \mathbb{R}$ for each $m \in \mathbb{N}$.

I suggest you proceed as follows.

(i) Use the chain rule and other rules for differentiation to show that

\[
\mathbb{R} \sim \{0\} \subset \text{dmn } \phi^{(m)} \text{ for each } m \in \mathbb{N}.
\]

(ii) Show by induction that there is for each $m \in \mathbb{N}$ a polynomial function $p_m : \mathbb{R} \to \mathbb{R}$ such that

\[
\phi^{(m)}(x) = e^{-\frac{1}{x}}p_m(x) \text{ whenever } x > 0.
\]

(iii) Show that

\[
\lim_{x \to 0} e^{-\frac{1}{x}} \frac{1}{x^N} = 0 \text{ whenever } N \in \mathbb{N}.
\]

(iv) Use (ii) and (iii) to show that

\[
\lim_{x \to 0} \phi^{(m)}(x) = 0
\]
for any $m \in \mathbb{N}$.

(v) Use 1. above to show that $0 \in \text{dmn } \phi^{(m)}$ and $\phi^{(m)}(0) = 0$ for any $m \in \mathbb{N}$. 

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1.4. Centered differences. Suppose $I$ is an open interval, $f : I \to \mathbb{R}$ and $f$ is three times differentiable at each point of $I$. Let

$$M = \sup\{|f^{(3)}(x)| : x \in I\}.$$ 

Use Taylor’s theorem to show that

$$\left| f(a + h) - f(a - h) - 2hf'(a) \right| \leq \frac{Mh^2}{3}.$$ 

1.5. Uniform convergence and differentiation. Suppose $I$ is an open interval and $f$ is a sequence of $\mathbb{R}$ valued functions on $I$ with the property that it and the sequence of derivatives converges uniformly on $I$ to $F$ and $G$, respectively. Show that $F$ is differentiable at each point of $I$ and that

$$F' = G.$$ 

**Hint.** Note that

$$\frac{F(x) - F(a)}{x - a} - G(a) = \frac{f_n(x) - f_n(a)}{x - a} - \frac{f'_n(a)}{x - a} + \left( \frac{(F - f_n)(x) - (F - f_n)(a)}{x - a} \right) + [f'_n(a) - G(a)]$$

and that

$$\frac{(F - f_n)(x) - (F - f_n)(a)}{x - a} = \lim_{m \to \infty} \frac{(f_m - f_n)(x) - (f_m - f_n)(a)}{x - a}$$

whenever $x, a \in I$, $x \neq a$ and $n \in \mathbb{N}$. Show that the second and third terms can be made small by making $n$ large independently of $a$ and $x$; to deal with the second term make use of the Mean Value Theorem.

**Bonus question; not really too hard.** Show that instead of supposing $f$ converges to $F$ uniformly it suffices to assume that, for some $a \in I$, $f_n(a) \to F(a)$ as $n \to \infty$. 