1. Homework Eight. Due Friday, October 23.

1.1. An exercise on difference quotients. Suppose I is an open interval, $a \in I$, $f : I \to \mathbb{R}$ and f is differentiable at a.

Show that for each $\epsilon > 0$ there is $\delta > 0$ such that

$$a - \delta < x < a \text{ and } a < y < a + \delta \;\Rightarrow \left| \frac{f(y) - f(x)}{y - x} - f'(a) \right| < \epsilon.$$

Show by example that it is not necessarily the case that

$$a < x < a + \delta \text{ and } a < y < a + \delta \;\Rightarrow \left| \frac{f(y) - f(x)}{y - x} - f'(a) \right| < \epsilon.$$

1.2. An exercise on differentiation. Suppose I is an open interval, $a \in I$, $f : I \to \mathbb{R}$, f is differentiable at each point of $I \sim \{a\}$, f is continous at a and

$$\lim_{x \to a} f'(x) \;=\; L$$

for some $L \in \mathbb{R}$. Prove that f is differentiable at a and $f'(a) = L$.

1.3. A very useful example. We define

$$\phi : \mathbb{R} \to \mathbb{R}$$

by requiring that

$$\phi(x) = \begin{cases} 0 & \text{if } x \le 0, \\ e^{-\frac{1}{x}} & \text{if } x > 0. \end{cases}$$

Show that

$$\mathbf{dmn}\, \phi^{(m)} = \mathbb{R} \quad \text{for each } m \in \mathbb{N}.$$

I suggest you proceed as follows.

(i) Use the chain rule and other rules for differentiation to show that

$$\mathbb{R} \sim \{0\} \subset \mathbf{dmn}\, \phi^{(m)} \quad \text{for each } m \in \mathbb{N}.$$

(ii) Show by induction that there is for each $m \in \mathbb{N}$ a polynomial function $p_m : \mathbb{R} \to \mathbb{R}$ such that

$$\phi^{(m)}(x) = e^{-\frac{1}{x}} p_m(x) \quad \text{whenever } x > 0.$$

(iii) Show that

$$\lim_{x \downarrow 0} e^{-\frac{1}{x}} \frac{1}{x^N} = 0 \quad \text{whenever } N \in \mathbb{N}.$$

(iv) Use (ii) and (iii) to show that

$$\lim_{x \to 0} \phi^{(m)}(x) = 0$$

for any $m \in \mathbb{N}$.

(v) Use 1. above to show that $0 \in \mathbf{dmn}\, \phi^{(m)}$ and $\phi^{(m)}(0) = 0$ for any $m \in \mathbb{N}$.

1.4. **Centered differences.** Suppose I is an open interval, $f : I \to \mathbb{R}$ and f is three times differentiable at each point of I. Let

$$M = \sup\{|f^{(3)}(x)| : x \in I\}.$$

Use Taylor's theorem to show that

$$\left| \frac{f(a+h) - f(a-h)}{2h} - f'(a) \right| \le \frac{Mh^2}{3}.$$

1.5. **Uniform convergence and differentiation.** Suppose I is an open interval and f is a sequence of $\mathbb{R}$ valued functions on I with the property that it and the sequence of derivatives converges uniformly on I to F and G, respectively. Show that F is differentiable at each point of I and that

$$F' = G.$$

Hint. Note that

$$\frac{F(x) - F(a)}{x - a} - G(a) = \left[\frac{f_n(x) - f_n(a)}{x - a} - f_n'(a) \right] + \left[\frac{(F - f_n)(x) - (F - f_n)(a)}{x - a} \right] + [f_n'(a) - G(a)]$$

and that

$$\frac{(F - f_n)(x) - (F - f_n)(a)}{x - a} = \lim_{m \to \infty} \frac{(f_m - f_n)(x) - (f_m - f_n)(a)}{x - a}$$

whenever $x, a \in I$, $x \neq a$ and $n \in \mathbb{N}$. Show that the second and third terms can be made small by making n large independently of a and x; to deal with the second term make use of the Mean Value Theorem.

Bonus question; not really too hard. Show that instead of supposing f converges to F uniformly it suffices to assume that, for some $a \in I$, $f_n(a) \to F(a)$ as $n \to \infty$.