Homework Five. Due Monday, September 28, 2009

1. A basic property of sup and inf.

Exercise 1.1. Suppose A is a nonempty subset of $\mathbb{R}$. Show that if A has an upper bound then $\sup A \in \mathbf{cl}\, A$. Show that if A has a lower bound then $\inf A \in \mathbf{cl}\, A$.

2. Limits and continuity.

Exercise 2.1. Suppose $A \subset \mathbb{R}^n$, $f : A \to \mathbb{R}^m$. Prove that f is continuous if and only if for each $a \in A$ and each $\epsilon > 0$ there is $\delta > 0$ such that

$$x \in A \text{ and } |x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon.$$

Hint: You can do this from scratch if you like but it is a straightforward consequence of Theorems in **Topological spaces.**

Exercise 2.2. Prove Theorems 1.22 and 1.23 in **Topological spaces**.

3. Compactness.

Exercise 3.1. Suppose X and Y are topological spaces, A is a compact subset of X and $f : A \to Y$ is continuous and X is compact. Prove that $f[A]$ is compact.

Show by counterexample that $f[A]$ need not be compact if A is not compact.

Exercise 3.2. (Difficult, but not terribly so.) Show that $K = [0,1] \times [0,1]$ is compact. (You may use the fact that $[a,b]$ is compact whenever $-\infty < a < b < \infty$ and anything that precedes that but nothing else unless you come up with it.)

Hint: Suppose $\mathcal{U}$ is an open covering of K and $y \in [0,1]$. First show that we may assume without loss of generality that each member of $\mathcal{U}$ is the product $I \times J$ of open intervals I and J.

Using the fact that $[0,1]$ is compact show that there are for each $y \in [0,1]$ a positive number ϵ_y; a positive integer N; and open intervals I_i and J_i, $i = 1, \ldots, N$, such that

$$I_i \times J_i \in \mathcal{U}, \quad i = 1, \ldots, N,$$

and

$$[0,1] \times (y - \epsilon_y, y + \epsilon_y) \subset \bigcup_{i=1}^{N} I_i \times J_i.$$

Remark 3.1. One can easily soup this up to show that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. We'll shortly prove a more general theorem.