Homework Four. Due Monday, September 21, 2009

1. The topology of $\mathbb{R}$.

Exercise 1.1. Prove using nothing but the definition that $(0,1) \times [0,1]$ is not a compact subset of $\mathbb{R}^2$.

Exercise 1.2. Prove using nothing but the definition that $\{0,1\} \times \mathbb{R}$ is not connected a connected subset of $\mathbb{R}^2$.

Exercise 1.3. Prove that

$$\mathbf{cl}\,\mathbf{Q} = \mathbb{R}, \quad \mathbf{cl}\,(\mathbb{R} \sim \mathbf{Q}) = \mathbb{R}, \quad \mathbf{int}\,\mathbf{Q} = \emptyset, \quad \mathbf{int}\,(\mathbb{R} \sim \mathbf{Q}) = \emptyset.$$

Hint: You will need to use that fact that if $a, b \in \mathbb{R}$ and $a < b$ then there is $q \in \mathbb{Q}$ such that $a < q < b$. Also, if you don't use the Theorems in the notes efficiently you will need to work much harder.

Exercise 1.4. Suppose $x \in \mathbb{R} \sim \mathbf{Q}$ and let

$$A = \{m + nx : m, n \in \mathbf{Z}\}.$$

Prove that $\mathbf{cl}\,A = \mathbb{R}$. (This is *very* tricky.) Big hint: It will suffice to show that for each $\epsilon > 0$ there is $a \in A$ such that $0 < a < \epsilon$.

2. The topology of $\mathbb{R}^n$.

Do Exercise 1.1 from **Topological spaces**.

3. Normed linear spaces.

Suppose X is a vector space. We say a function

$$|\cdot| : X \to [0, \infty)$$

is a **norm (on X)** if

(i) $|cx| = |c||x|$ whenever $c \in \mathbb{R}$ and $x \in X$;
(ii) $|x + y| \leq |x| + |y|$ whenever $x, y \in X$;
(iii) if $x \in X$ and $|x| = 0$ then $x = 0$.

Exercise 3.1. Suppose $|\cdot|$ is a norm on X. Declare a subset U of X to be open if for each $a \in U$ there is $\epsilon > 0$ such that

$$\{x \in X : |x - a| < \epsilon\} \subset U.$$

Show that the open subsets of X are a topology on X which is Hausdorff. (Note that $|x \pm y| \geq |x| - |y|$ whenever $x, y \in X$. What you did to do Exercise 1.1 above should carry over directly to the present situation.)

Exercise 3.2. Prove that the closure of a linear subspace of a normed vector space is a linear subspace.

Definition 3.1. A subset C of a vector space is **convex** if

$$a, b \in C \;\Rightarrow\; \{(1-t)a + tb : 0 < t < 1\} \subset C.$$

Exercise 3.3. Prove that the interior and closure of a convex subset of a normed vector space are convex.

4. Product topologies.

Suppose A is a nonempty set and X is a function with domain A such that

$$X_a \text{ is a nonempty topological space for each } a \in A.$$

Let

$$\prod_{a\in A} X_a \quad \text{or, more compactly,} \quad \prod X$$

be the set of functions x with domain A such that

$$x_a \in X_a \quad \text{whenever } a \in A.$$

(It follows from the Axiom of Choice that $\prod_{a\in A} X_a$ is nonempty.) Think of a member of $\prod X$ as an A-tuple whose a-th component is a member of X_a for each $a \in A$. For each $a \in A$ let

$$p_a : \prod X \to X_a$$

be such that

$$p_a(x) = x_a \quad \text{whenever } x \in \prod X.$$

Let

$$\mathcal{B}$$

be the family of subsets V of $\prod X$ such that there are a finite subset F of A and for each $a \in F$ an open subset U_a of X_a such that

$$V = \bigcap_{a\in F} p_a^{-1}[U_a].$$

Exercise 4.1. Show that

$$\mathcal{T} = \{\cup\mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$$

is a topology for $\prod X$. In fact, it should be obvious that $\mathcal{T}$ is closed with respect to unions so it will suffice to show that $\mathcal{B}$ is closed with respect to finite intersections. (Not surprisingly, this topology is called the **product topology**.)