Hermitian inner products.

Suppose $V$ is vector space over $\mathbb{C}$ and $(\cdot, \cdot)$ is a Hermitian inner product on $V$. This means, by definition, that

$$(\cdot, \cdot) : V \times V \to \mathbb{C}$$

and that the following four conditions hold:

(i) $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ whenever $v_1, v_2, w \in V$;

(ii) $(cv, w) = c(v, w)$ whenever $c \in \mathbb{C}$ and $v, w \in V$;

(iii) $(w, v) = \overline{(v, w)}$ whenever $v, w \in V$;

(iv) $(v, v)$ is a positive real number for any $v \in V \sim \{0\}$.

These conditions imply that

(v) $(v, w_1 + w_2) = (v, w_1) + (v, w_2)$ whenever $v, w_1, w_2 \in V$;

(vi) $(v, cw) = \overline{c}(v, w)$ whenever $c \in \mathbb{C}$ and $v, w \in V$;

(vii) $(0, v) = 0 = (v, 0)$ for any $v \in V$.

In view of (iv) and (vii) we may set

$$||v|| = \sqrt{(v, v)} \quad \text{for } v \in V$$

and note that

(viii) $||v|| = 0 \iff v = 0$.

We call $||v||$ the norm of $v$. Note that

(ix) $||cv|| = ||c|| ||v||$ whenever $c \in \mathbb{C}$ and $v \in V$.

Suppose

$$A : V \times V \to \mathbb{R} \quad \text{and} \quad B : V \times V \to \mathbb{R}$$

are such that

$$(1) \quad (v, w) = A(v, w) + iB(v, w) \quad \text{whenever } v, w \in V.$$

One easily verifies that

(i) $A$ and $B$ are bilinear over $\mathbb{R}$;

(ii) $A$ is symmetric and positive definite;

(iii) $B$ is antisymmetric;

(iv) $A(iv, iw) = A(v, w)$ whenever $v, w \in V$;
(v) \( B(v, w) = -A(iv, w) \) whenever \( v, w \in V \).

Conversely, given \( A : V \times V \to \mathbb{R} \) which is bilinear over \( \mathbb{R} \) and which is positive definite symmetric, letting \( B \) be as in (v) and let \( (\cdot, \cdot) \) be as in (1) we find that \( (\cdot, \cdot) \) is a Hermitian inner product on \( V \). The interested reader might write down conditions on \( B \) which allow one to construct \( A \) and \( (\cdot, \cdot) \) as well.

Example 0.1. Let

\[
(z, w) = \sum_{j=1}^{n} z_j \overline{w_j} \quad \text{for } z, w \in \mathbb{C}^n.
\]

The \( (\cdot, \cdot) \) is easily seen to be a Hermitian inner product, called the **standard (Hermitian) inner product**, on \( \mathbb{C}^n \).

Example 0.2. Suppose \( -\infty < a < b < \infty \) and \( \mathcal{H} \) is the vector space of complex valued square integrable functions on \( [a, b] \). You may object that I haven’t told you what “square integrable” means. Now I will. Sort of. To say \( f : [a, b] \to \mathbb{R} \) is **square integrable** means that \( f \) is Lebesgue measurable and that

\[
\int_a^b |f(x)|^2 \, dx < \infty;
\]

of course I haven’t told you what “Lebesgue measurable” means and I haven’t told you what \( \int_a^b \) means, but I will in the very near future. For the time being just think of whatever notion of integration you’re familiar with.

Note that

\[
\int_a^b f(x) \, dx = \int_a^b \Re f(x) \, dx + i \int_a^b \Im f(x) \, dx
\]

whenever \( f \in \mathcal{H} \).

Let

\[
(f, g) = \int_a^b f(x) \overline{g(x)} \, dx \quad \text{whenever } f, g \in \mathcal{H}.
\]

You should object at this point that the integral may not exist. We will show shortly that it does. One easily verifies that (i)-(iii) of the properties of an inner product hold and that (iv) **almost** holds in the sense that for any \( f \in \mathcal{F} \) we have

\[
(f, f) = \int_a^b |f(x)|^2 \, dx \geq 0
\]

with equality only if \( \{ x \in [a, b] : f(x) = 0 \} \) has zero Lebesgue measure (whatever that means). In particular, if \( f \) is continuous and \( (f, f) = 0 \) then \( f(x) = 0 \) for all \( x \in [a, b] \).

This Example is like Example One in that one can think of \( f \in \mathcal{H} \) as a an infinite-tuple with the continuous index \( x \in [a, b] \).

Henceforth \( V \) is a Hermitian inner product space.

The following simple Proposition is indispensable.

**Proposition 0.1.** Suppose \( v, w \in V \). Then

\[
\|v + w\|^2 = \|v\|^2 + 2\Re(v, w) + \|w\|^2.
\]
Proof. We have
\[ \|v + w\|^2 = (v + w, v + w) \]
\[ = (v, v) + (v, w) + (w, v) + (w, w) \]
\[ = (v, v) + (v, w) + (w, v) + (w, w) \]
\[ = \|v\|^2 + 2\Re(v, w) + \|w\|^2. \]

Corollary 0.1 (The Parallelogram Law). We have
\[ \|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2) . \]

Proof. Look at it.

Here is an absolutely fundamental consequence of the Parallelogram Law.

**Theorem 0.1.** Suppose $V$ is complete with respect to $\| \cdot \|$ and $C$ is a nonempty closed convex subset of $V$. Then there is a unique point $c \in C$ such that
\[ \|c\| \leq \|v\| \quad \text{whenever } v \in C. \]

**Remark 0.1.** Draw a picture.

Proof. Let
\[ d = \inf\{\|v\| : v \in C\} \]
and let
\[ \mathcal{C} = \{C \cap B^d(r) : d < r < \infty\}. \]

Note that $\mathcal{C}$ is a nonempty nested family of nonempty closed subsets of $V$.

Suppose $C \in \mathcal{C}$, $d < r < \infty$ and $v, w \in C$. Because $C$ is convex we have
\[ \frac{1}{2}(v + w) \in C \cap B^d(R) \]
so
\[ \frac{1}{4}\|v + w\|^2 = \frac{1}{2}(v + w))^2 \geq d^2. \]

Thus, by the Parallelogram Law,
\[ \frac{1}{4}\|v - w\|^2 = \frac{1}{2} (\|v\|^2 + \|w\|^2) - \frac{1}{4}\|v + w\|^2 \leq r^2 - d^2. \]

It follows that
\[ \inf\{\text{diam}C \cap B^d(r) : d < r < \infty\} = 0. \]

By completeness there is a point $c \in V$ such that
\[ \{c\} = \cap \mathcal{C}. \]

\[ \square \]

**Corollary 0.2.** Suppose $U$ is a closed linear subspace of $V$ and $v \in V$. Then there is a unique $u \in U$ such that
\[ \|v - u\| \leq \|v - u'\| \quad \text{whenever } u' \in U. \]

**Remark 0.2.** Draw a picture.

**Remark 0.3.** We will show very shortly that any finite dimensional subspace of $V$ is closed.
Proof. Let $C = v - U$ and note that $C$ is a nonempty closed convex subset of $V$. (Of course $-U = U$ since $U$ is a linear subspace of $U$, but this representation of $C$ is more convenient for our purposes.) By virtue of the preceding Theorem there is a unique $u \in U$ such that

$$||v - u|| \leq ||v - u'||$$ whenever $u' \in U$.

\[\square\]

Theorem 0.2 (The Cauchy-Schwartz Inequality.). Suppose $v, w \in V$. Then

$$(v, w) \leq ||v|| ||w||$$

with equality only if $\{v, w\}$ is dependent.

Proof. If $w = 0$ the assertion holds trivially so let us suppose $w \neq 0$. For any $c \in \mathbb{C}$ we have

$$0 \leq ||v + cw||^2 = ||v||^2 + 2\Re(v, cw) + ||cw||^2 = ||v||^2 + 2\Re(v, w) + |c|^2 ||w||^2.$$

Letting

$$c = \frac{(v, w)}{||w||^2}$$

we find that

$$0 \leq ||v||^2 - \frac{(v, w)^2}{||w||^2}$$

with equality only if $||v + cw|| = 0$ in which case $v + cw = 0$ so $v = -cw$. \[\square\]

Corollary 0.3. Suppose $a$ and $b$ are sequences of complex numbers. Then

$$\sum_{n=0}^{\infty} |a_n b_n| \leq \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} |b_n|^2 \right)^{1/2}.$$

Proof. For any nonnegative integer $N$ apply the Cauchy-Schwartz inequality with $(\cdot, \cdot)$ equal the standard inner product on $\mathbb{C}^N$,

$$v = (a_0, \ldots, a_N) \quad \text{and} \quad w = (b_0, \ldots, b_N)$$

and then let $N \to \infty$. \[\square\]

Theorem 0.3 (The Triangle Inequality.). Suppose $v, w \in V$. Then

$$||v + w|| \leq ||v|| + ||w||$$

with equality only if either $v$ is a nonnegative multiple of $w$ or $w$ is a nonnegative multiple of $v$.

Proof. Using the Cauchy-Schwartz Inequality we find that

$$||v + w||^2 = ||v||^2 + 2\Re(v, w) + ||w||^2 \leq ||v||^2 + 2||v|| ||w|| + ||w||^2 = (||v|| + ||w||)^2.$$

Suppose equality holds. In case $v = 0$ then $v = 0w$ so suppose $v \neq 0$. Since $|(v, w)| \geq \Re(v, w) = ||v|| ||w||$ we infer from the Cauchy-Schwartz Inequality that $w = cw$ for some $c \in \mathbb{C}$. Thus

$$|1 + c||v|| = ||(1 + c)v|| = ||v + cw|| = ||v|| + ||cw|| = (1 + |c|)||v||$$
from which we infer that
\[ 1 + 2\Re c + |c|^2 = |1 + c|^2 = (1 + |c|)^2 = 1 + 2|c| + |c|^2 \]
which implies that \( c \) is a nonnegative real number. \( \square \)

**Definition 0.1.** Suppose \( U \) is a linear subspace of \( V \). We let
\[ U^\perp = \{ v \in V : (u, v) = 0 \text{ for all } u \in U \} \]
and note that \( U^\perp \) is a linear subspace of \( V \). It follows directly from (iv) that
\[ U \cap U^\perp = \{ 0 \} \]

**Proposition 0.2.** Suppose \( U \) is a linear subspace of \( V \). Then
\[ U \subset U^{\perp \perp} \]
and \( U^\perp \) is closed.

**Proof.** The first assertion is an immediate consequence of the definition of \( U^\perp \). The second follows because \( U^\perp \) is the intersection of the closed sets
\[ \{ v \in V : (u, v) = 0 \} \]
corresponding to \( u \in U \); These sets are closed because \( V \ni v \mapsto (u, v) \) is continuous by virtue of the Cauchy-Schwartz Inequality. \( \square \)

**Orthogonal projections.**

Henceforth \( U \) is closed linear subspace of \( V \).

**Definition 0.2.** Keeping in mind the foregoing, we define
\[ P : V \to U \]
by requiring that
\[ ||v - Pv|| \leq ||v - u'|| \quad \text{whenever } u' \in U. \]

That is, \( Pv \) is the closest point in \( U \) to \( v \). We call \( P \) **orthogonal projection** of \( V \) **onto** \( U \). Note that \( Pu = u \) whenever \( u \in U \). Thus
\[ \text{rng } P = U \quad \text{and} \quad P \circ P = P. \]

Keeping in mind that \( U^\perp \) is a closed linear subspace of \( V \) we let
\[ P^{\perp} \]
be orthogonal projection of \( V \) onto \( U^\perp \).

**Theorem 0.4.** Suppose \( W \) is a linear subspace of \( V \) and
\[ Q : V \to W \]
is such that
\[ ||w - Qv|| \leq ||v - w|| \quad \text{whenever } v \in V \text{ and } w \in W. \]
Then \( W \) is closed and \( Q \) is orthogonal projection of \( V \) onto \( W \).
Proof. Suppose \( \tilde{w} \in \text{cl}W \) and \( \epsilon > 0 \). Choose \( w \in W \) such that \( ||\tilde{w} - w|| \leq \epsilon \). Then

\[
||\tilde{w} - Q\tilde{w}|| \leq ||\tilde{w} - w|| \leq \epsilon.
\]

Owing to the arbitrariness of \( \epsilon \) we infer that \( ||Q\tilde{w} - w|| = 0 \) so \( w = Q\tilde{w} \in W \) and \( \text{cl}W \subset W \). \( \square \)

**Theorem 0.5.** We have

\[
u = \Pi v \iff v - u \in U^{\perp} \quad \text{whenever } u \in U \text{ and } v \in V.
\]

**Proof.** Suppose \( u \in U \) and \( v \in V \). For each \((t, u') \in \mathbb{R} \times U\) let

\[
f(t, u') = ||(v - u) + tu'||^2
\]

and note that

\[
f(t, u') = ||v - u||^2 + 2t\Re(v - u, u') + t^2||u'||^2.
\]

Suppose \( u = \Pi v \). Then \( f(0, u') \leq f(t, u') \) whenever \((t, u') \in \mathbb{R} \times U\). Thus \( v - u \in U^{\perp} \).

Suppose \( v - u \in U^{\perp} \). Then

\[
||v - u||^2 = f(0, u' - u) \leq f(1, u' - u) = ||v - u'||^2
\]

so \( u = \Pi v \). \( \square \)

**Corollary 0.4.** \( \Pi \) is linear.

**Proof.** Suppose \( v \in V \) and \( c \in \mathbb{C} \). Then \( c\Pi v = \Pi(cv) \in U^{\perp} \) so \( \Pi(cv) = c\Pi v \). Suppose \( v_1, v_2 \in V \). then \( \Pi v_1 + \Pi v_2 \in U^{\perp} \) and \((v_1 + v_2) - (\Pi v_1 + \Pi v_2) = (v_1 - \Pi v_1) + (v_2 - \Pi v_2) \in U^{\perp} \) so \( \Pi(v_1 + v_2) = \Pi v_1 + \Pi v_2 \). \( \square \)

**Corollary 0.5.** Suppose \( v \in V \). Then

(i) \( v = \Pi v + \Pi^\perp v \) and

(ii) \( ||v||^2 = ||\Pi v||^2 + ||\Pi^\perp v||^2 \).

**Proof.** We have \( v - \Pi v \in U^{\perp} \) by the preceding Theorem and

\[
v - (v - \Pi v) = \Pi v \in U \subset U^{\perp}
\]

so, again by the preceding Theorem only with \( U \) replaced by \( U^{\perp} \) we find that \( \Pi^\perp v = v - \Pi v \). It follows that

\[
||v||^2 = ||\Pi v + \Pi^\perp v||^2 = ||\Pi v||^2 + 2\Re(\Pi v, \Pi^\perp v) + ||\Pi^\perp v||^2 = ||\Pi v||^2 + ||\Pi^\perp v||^2.
\]

**Corollary 0.6.** We have

\[
U^{\perp\perp} = U
\]

and

\[
(\Pi v, w) = (v, \Pi w) \quad \text{whenever } v, w \in V.
\]
Proof. Let $P$ and $P^\perp$ be orthogonal projection of $V$ onto $U$ and $U^\perp$, respectively. By the preceding Theorem with $U$ replaced by $U^\perp$ we find that orthogonal projection of $V$ onto $U^\perp$ carries $v \in V$ to $v - P^\perp v = Pv$. Thus $U = U^\perp$.

Suppose $v, w \in V$. Then

\[(Pv, w) = (Pv, Pw + P^\perp w) = (Pv, Pw) = (Pv + P^\perp v, Pw) = (v, Pw).\]

\[\square\]

**Definition 0.3.** We say a subset $A$ of $V$ is **orthonormal** if whenever $v, w \in A$ we have

\[(v, w) = \begin{cases} 1 & \text{if } v = w; \\ 0 & \text{if } v \neq w. \end{cases}\]

**Exercise 0.1.** Show that any orthonormal set is independent.

**The Gram-Schmidt Process.** Suppose $\tilde{u} \in V \sim U, \tilde{U} = \{u + c\tilde{u} : c \in \mathbb{C}\}$ and

\[\tilde{P}v = Pv + \frac{(v, P^\perp \tilde{u})}{||P^\perp \tilde{u}||^2} P^\perp \tilde{u}\]

whenever $v \in V$.

Then $\tilde{U}$ is closed and $\tilde{P}$ is orthogonal projection on $\tilde{U}$.

**Proof.** Easy exercise for the reader. \[\square\]

**Remark 0.4.** If $U = \{0\}$ then $P = 0$ so

\[\tilde{P}(v) = \frac{(v, \tilde{u})}{||\tilde{u}||^2} \tilde{u}\]

and $\tilde{P}$ is orthogonal projection on the line $\{c\tilde{u} : c \in \mathbb{C}\}$.

**Corollary 0.7.** Any finite dimensional subspace of $V$ is closed and has an orthonormal basis.

**Proof.** Induct on the dimension of the subspace and use the Gram-Schmidt Process to carry out the inductive step. \[\square\]

**Proposition 0.3.** Suppose $U$ is finite dimensional and $B$ is an orthonormal basis for $U$. Then

\[Pv = \sum_{u \in B} (v, u)u\]

and

\[||Pv||^2 = \sum_{u \in B} |(v, u)|^2\]

whenever $v \in V$.

**Proof.** Let

\[Lv = \sum_{u \in B} (v, u)u\]

for $v \in V$. \[\square\]
Suppose $v \in V$ and $\tilde{u} \in B$. The
\[
(v - Lv, \tilde{u}) = (v - \sum_{u \in B} (v, u)u, \tilde{u})
= (v, \tilde{u}) - \sum_{u \in B} (v, u)(u, \tilde{u})
= (v, \tilde{u}) - (v, \tilde{u})
= 0
\]
which, as $B$ is a basis for $U$, implies that $v - Lv \in U^\perp$; thus $P = L$.

Finally, if $v \in V$ we have
\[
||Lv||^2 = (\sum_{u \in B} (v, u)u, \sum_{\tilde{u} \in B} (v, \tilde{u})\tilde{u})
= \sum_{u \in B, \tilde{u} \in B} (v, u)(v, \tilde{u})(u, \tilde{u})
= \sum_{u \in B} |(u, v)|^2.
\]

\[\square\]

Hilbert space.

Let $X$ be a set and let
\[
H_X = \{u \in \mathcal{C}^X : \sum_X |u|^2 < \infty\}.
\]

Proposition 0.4. Suppose $u, v \in H_X$. Then
\[
\sum_X |uv| < \infty.
\]

Proof. Suppose $F$ is a finite subset of $X$. The Cauchy-Schwartz Inequality implies that
\[
\left(\sum_F |uv|\right)^2 \leq \left(\sum_F |u|^2\right) \left(\sum_F |v|^2\right) \leq \left(\sum_X |u|^2\right) \left(\sum_X |v|^2\right) < \infty.
\]

\[\square\]

Definition 0.4. Keeping in mind the previous Proposition we let
\[
(u, v) = \sum_X u\overline{v} \quad \text{whenever } u, v \in H_X.
\]

One easily verifies that $(\cdot, \cdot)$ is a Hermitian inner product on $H_X$.

Definition 0.5. For each subset $A$ of $X$ let
\[
H^A
\]

Theorem 0.6. $H_X$ is complete.
Proof. Let $\mathcal{C}$ be a nonempty nested family of nonempty closed subsets of $H_X$ such that $\inf \{\text{diam } C : C \in \mathcal{C} \} = 0$. For each $C \in \mathcal{C}$ let
\[ b_C = \sup \{ ||v|| : v \in C \}. \]
By the triangle inequality there are $B \in [0, \infty)$ and $C_0 \in \mathcal{C}$ such that $b_{C_0} \leq B$.

Note that $b_C \leq b_{C_0}$ whenever $C \subseteq \mathcal{C}$ and $C \subseteq C_0$.

For each $x \in X$ let $C_x = \overline{\{ u(x) : u \in C \}}$ for each $C \in \mathcal{C}$, note that
\[ \text{diam } C_x \leq \text{diam } C \quad \text{for each } C \in \mathcal{C}, \]
and let
\[ \mathcal{C}_x = \{ C_x : C \in \mathcal{C} \}. \]
For each $x \in X$ the family $\mathcal{C}_x$ is a nonempty nested family of nonempty closed subsets of $C$ and $\inf \{\text{diam } C_x : C \in \mathcal{C} \} = 0$. Since $C$ is complete there is one and only one $u \in C^X$ such that
\[ u(x) \in \cap \mathcal{C}_x \quad \text{whenever } x \in X. \]

Suppose $F$ is a finite subset of $X$. Choose $C \in \mathcal{C}$ such that $C \subseteq C_0$ and $|F|\text{diam } C^2 \leq 1$. Suppose $v \in C$. We infer from the Triangle Inequality that
\[ \left( \sum_F |u|^2 \right)^{1/2} \leq \left( \sum_F |u - v|^2 \right)^{1/2} + \left( \sum_F |v|^2 \right)^{1/2} \leq \sqrt{|F|} \text{max}\{\text{diam } C_x : x \in F\} + ||v||^2 \leq \sqrt{|F|} \text{max}\{\text{diam } C_x : x \in F\} + ||v||^2 \]
It follows that
\[ u \in H_X. \]

Suppose $\epsilon > 0$, and
\[ \left( \sum_F |u - v|^2 \right)^{1/2} \leq \left( \sum_F |u - v|^2 \right)^{1/2} + \left( \sum_F |v|^2 \right)^{1/2} \leq \sqrt{|F|} \text{max}\{\text{diam } C_x : x \in F\} + ||v||^2 \]
\[ \square \]