The Brouwer Fixed Point Theorem.

Fix a positive integer \( n \) and let \( D^n = \{ x \in \mathbb{R}^n : |x| \leq 1 \} \). Our goal is to prove

**The Brouwer Fixed Point Theorem.** Suppose

\[
f : D^n \to D^n
\]

is continuous. Then \( f \) has a **fixed point**; that is, there is \( a \in D^n \) such that \( f(a) = a \).

This will follow quickly from the following

**Theorem. You can’t retract the ball to its boundary.** There exists no continuous retraction

\[
r : D^n \to S^{n-1}.
\]

(We say \( r : X \to Y \) is a retraction if \( Y \subset X \) and \( r(y) = y \) whenever \( y \in Y \).)

Indeed, suppose \( f : D^n \to D^n \) is continuous but has no fixed point. For each \( x \in D^n \) let \( r(x) \) be the point in \( S^{n-1} \) determined by the requirement that

\[
r(x) = f(x) + \lambda(x - f(x))
\]

for some positive real number \( \lambda \). We leave it to the reader to verify that \( r \) would be a continuous retraction of the ball \( D^n \) to its boundary \( S^{n-1} \).

**The proof that you can’t retract the ball to its boundary.** Suppose, to the contrary, that \( r \) continuously retracts the ball \( D^n \) to its boundary \( S^{n-1} \).

**Step One.** Choose \( \epsilon \in (0, 1/2) \). Using \( r \) we construct smooth function \( s : \mathbb{R}^n \to \mathbb{R}^n \sim \{0\} \) such that

\[
s(x) = x \quad \text{whenever for } |x| > 1 - \epsilon.
\]

To this end we define the function \( R : \mathbb{R}^n \to \mathbb{R}^n \sim \{0\} \) by letting

\[
R(x) = \begin{cases} 
    r(\frac{1}{1-2\epsilon}x) & \text{if } |x| < 1 - 2\epsilon, \\
    x & \text{else}.
\end{cases}
\]

Note that \( R \) is continuous. Let \( \phi \) be a smooth even function on \( \mathbb{R}^n \) whose support is a subset \( U_\epsilon(0) \) and which satisfies \( \int \phi = 1 \). Let \( s = \phi \ast R \). Suppose \( |a| > 1 - \epsilon \). Then

\[
s(a) = \phi \ast R(a) = a - \int \phi(a-x)(a-x) \, dx = a
\]

since \( y \mapsto \phi(y) y \) is odd.

**Step Two.** Let \( s \) be as in Part One. Let \( \Omega \) be the solid angle form on \( \mathbb{R}^n \sim \{0\} \). Evidently, \( s \# \Omega = \Omega \) on \( \{x : |x| > 1 - \epsilon\} \). Keeping in mind that \( \Omega \) is closed we use Stokes’ Theorem to calculate

\[
0 \neq \text{area } S^{n-1} = \int_{S^{n-1}} \Omega = \int_{S^{n-1}} s \# \Omega = \int_{\partial D^n} s \# \Omega = \int_{D^n} ds \# \Omega = \int_{D^n} s \# d\Omega = 0,
\]

\footnote{Draw a picture! The point here is that \( |f(x) + \lambda(x - f(x))|^2 = 1 \) is a quadratic equation for \( \lambda \) which has exactly one nonnegative solution.}