The Brouwer Fixed Point Theorem.

Fix a positive integer n and let $\mathbf{D}^n = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| \le 1\}$. Our goal is to prove

The Brouwer Fixed Point Theorem. Suppose

$$f : \mathbf{D}^n \to \mathbf{D}^n$$

is continuous. Then f has a **fixed point**; that is, there is $\mathbf{a} \in \mathbf{D}^n$ such that $f(\mathbf{a}) = \mathbf{a}$.

This will follow quickly from the following

Theorem. You can't retract the ball to its boundary. There exists no continuous retraction

$$r : \mathbf{D}^n \to \mathbf{S}^{n-1}.$$

(We say $r : X \to Y$ is a **retraction** if $Y \subset X$ and $r(y) = y$ whenever $y \in Y$.)

Indeed, suppose $f : \mathbf{D}^n \to \mathbf{D}^n$ is continuous but has no fixed point. For each $\mathbf{x} \in \mathbf{D}^n$ let $r(\mathbf{x})$ be the point in $\mathbf{S}^{n-1}$ determined by the requirement that

$$r(\mathbf{x}) = f(\mathbf{x}) + \lambda(\mathbf{x} - f(\mathbf{x}))$$

for some positive real number λ.[1] We leave it to the reader to verify that r would be a continuous retraction of the ball $\mathbf{D}^n$ to its boundary $\mathbf{S}^{n-1}$.

The proof that you can't retract the ball to its boundary. Suppose, to the contrary, that r continuously retracts the ball $\mathbf{D}^n$ to its boundary $\mathbf{S}^{n-1}$.

Step One. Choose $\epsilon \in (0, 1/2)$. Using r we construct smooth function $s : \mathbf{R}^n \to \mathbf{R}^n \sim \{\mathbf{0}\}$ such that

$$s(\mathbf{x}) \; = \; \mathbf{x} \quad \text{whenever for } |\mathbf{x}| > 1 - \epsilon.$$

To this end we define the function $R : \mathbf{R}^n \to \mathbf{R}^{\sim}\{\mathbf{0}\}$ by letting

$$R(\mathbf{x}) \; = \; \begin{cases} r(\frac{1}{1-2\epsilon}\mathbf{x}) & \text{if } |\mathbf{x}| < 1 - 2\epsilon, \\ \mathbf{x} & \text{else.} \end{cases}$$

Note that R is continuous. Let ϕ be a smooth even function on $\mathbf{R}^n$ whose support is a subset $\mathbf{U}_\epsilon(\mathbf{0})$ and which satisfies $\int \phi = 1$. Let $s = \phi * R$. Suppose $|\mathbf{a}| > 1 - \epsilon$. Then

$$s(\mathbf{a}) \; = \; \phi * R(\mathbf{a}) \; = \; \mathbf{a} - \int \phi(\mathbf{a} - \mathbf{x})(\mathbf{a} - \mathbf{x}) \, d\mathbf{x} \; = \; \mathbf{a}$$

since $\mathbf{y} \mapsto \phi(\mathbf{y})\mathbf{y}$ is odd.

Step Two. Let s be as in Part One. Let Ω be the solid angle form on $\mathbf{R}^n \sim \{\mathbf{0}\}$. Evidently, $s^\# \Omega = \Omega$ on $\{\mathbf{x} : |\mathbf{x}| > 1 - \epsilon\}$. Keeping in mind that Ω is closed we use Stokes' Theorem to calculate

$$0 \ne \mathbf{area}\, \mathbf{S}^{n-1} \; = \; \int_{\mathbf{S}^{n-1}} \Omega \; = \; \int_{\mathbf{S}^{n-1}} s^\# \Omega \; = \; \int_{\partial \mathbf{D}^n} s^\# \Omega \; = \; \int_{\mathbf{D}^n} d s^\# \Omega \; = \; \int_{\mathbf{D}^n} s^\# d\Omega \; = \; 0,$$

[1] Draw a picture! The point here is that $|f(\mathbf{x}) + \lambda(\mathbf{x} - f(\mathbf{x}))|^2 = 1$ is a quadratic equation for λ which has exactly one nonnegative solution.