1. The Fourier transform on $\mathbb{R}$.

Suppose

$$f : \mathbb{R} \to \mathbb{C}$$

is integrable; this means, by definition, $f \in \mathbf{Leb}_1$. For each $\xi \in \mathbb{R}$ we set

$$\hat{f}(\xi) = \int f(x) e^{-ix\xi}\, dx.$$

Evidently,

$$(0) \qquad |\hat{f}(\xi)| \le \int |f(x)|\, dx \quad \text{whenever } x \in \mathbb{R}.$$

This function is called the **Fourier transform of** f. Note that

$$(1) \qquad \lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$$

by the Riemann-Lebesgue Lemma.

Definition 1.1. Suppose $f : \mathbb{R} \to \mathbb{C}$. We let

$$\mathbf{spt}\, f = \mathbf{cl}\, \{f \neq 0\}$$

and call this closed set the **support of** f. We say f is **smooth** if $\mathbf{dmn}\, f^{(m)} = \mathbb{R}$ for all $m \in \mathbb{N}$.

We let

$$\mathcal{D}(\mathbb{R})$$

be the vector space of smooth $f : \mathbb{R} \to \mathbb{C}$ such that f is smooth and has compact support.

Theorem 1.1. Suppose $f \in \mathcal{D}(\mathbb{R})$. Then

$$\widehat{f'}(\xi) = i\xi \hat{f}(\xi) \quad \text{for } \xi \in \mathbb{R}.$$

Exercise 1.1. Prove this. Integrate by parts. Because f has compact support you may work on a bounded open interval which contains the support of f.

Corollary 1.1. Suppose $f \in \mathcal{D}(\mathbb{R})$. Then

$$(2) \qquad |\hat{f}(\xi)| \le \frac{1}{|\xi|^m} \int \left| f^{(m)}(x) \right| (x)\, dx \quad \text{for all } m \in \mathbb{N} \text{ and all } \xi \in \mathbb{R} \sim \{0\}.$$

Proof. This is an immediate consequence of the preceding Theorem. □

Corollary 1.2. Suppose $f \in \mathcal{D}(\mathbb{R})$. Then

$$\int |\hat{f}(\xi)|\, d\xi < \infty.$$

Exercise 1.2. Prove this.

1.1. **The Fourier Inversion Formula (Smooth version.)** Suppose
$$f \in \mathcal{D}(\mathbb{R}).$$
Then
$$f(x) = \frac{1}{2\pi}\int \hat{f}(\xi) e^{ix\xi}\, d\xi \quad \text{for } x \in \mathbb{R}. \tag{3}$$

The remainder of this subsection is devoted to a sketch of a proof of (3). We will use the Fourier Inversion Formula for periodic functions. The details are left as exercises for the reader.

Let M be a positive integer such that
$$\mathbf{spt}\, f \subset (-M, M).$$
Let
$$\mathbf{M} = \{m \in \mathbb{N} : m \geq M\}.$$
For each $m \in \mathbf{M}$ let $g_m : \mathbb{R} \to \mathbb{C}$ be determined by the requirement that
$$g_m(t) = f\left(\frac{mt}{\pi}\right) \quad \text{whenever } -\pi < t < \pi$$
and that g_m is 2π-periodic. Note that g_m is smooth.

Proposition 1.1. We have
$$\widehat{g_m}(n) = \frac{1}{\sqrt{2\pi}}\frac{\pi}{m}\hat{f}\left(\frac{n\pi}{m}\right) \quad \text{whenever } m \in \mathbf{M} \text{ and } n \in \mathbb{Z}.$$

Exercise 1.3. Prove this. Turn the crank.

Suppose $x \in \mathbb{R}$, $m \in \mathbf{M}$ and N is a positive integer. From the Fourier Inversion Formula in the periodic case we obtain
$$\begin{aligned} f(x) &= g_m\left(\frac{\pi x}{m}\right) \\ &= \sum_{n\in\mathbb{Z}} \widehat{g_m}(n) E_n\left(\frac{\pi x}{m}\right) \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\pi}{m}\hat{f}\left(\frac{n\pi}{m}\right) e^{\frac{in\pi x}{m}} \\ &= \frac{1}{2\pi}\left(T_1(x,m,N) + T_2(x,m,N) + T_3(x,m,N) + T_4(x)\right). \end{aligned}$$
where we have set
$$\begin{aligned} T_1(x,m,N) &= \sum_{n=-mN}^{mN-1} \int_{\frac{n\pi}{m}}^{\frac{(n+1)\pi}{m}} \left(\hat{f}\left(\frac{n\pi}{m}\right) e^{\frac{in\pi x}{m}} - \hat{f}(\xi) e^{ix\xi}\right) d\xi, \\ T_2(x,m,N) &= \left(\sum_{n<-mN} + \sum_{n\geq mN}\right) \frac{\pi}{m}\hat{f}\left(\frac{n\pi}{m}\right) e^{\frac{in\pi x}{m}}, \\ T_3(x,N) &= -\int_{|\xi|>M} \hat{f}(\xi) e^{ix\xi}\, d\xi, \\ T_4(x) &= \int \hat{f}(\xi) e^{ix\xi}\, d\xi. \end{aligned}$$

Let $\epsilon > 0$.

Exercise 1.4. Show that for any $N \in \mathbb{N}^+$ and any $\epsilon > 0$ there is $M \in \mathbf{M}$ such that

$$|T_1(x, m, N)| \leq \epsilon \quad \text{whenever } x \in \mathbb{R} \text{ and } m \geq M.$$

Hint: Note that

$$\mathbb{R} \ni \xi \mapsto \hat{f}(\xi) e^{ix\xi}$$

is uniformly continuous.

Exercise 1.5. Show that for any $\epsilon > 0$ there is $O \in \mathbb{N}^+$ such that

$$|T_2(x, m, N)| \leq \epsilon \quad \text{whenever any } x \in \mathbb{R},\ m \in \mathbf{M} \text{ and } N \geq O. \tag{4}$$

Hint: Use (2).

Exercise 1.6. Show that for any $\epsilon > 0$ there is $O \in \mathbb{N}^+$ such that

$$|T_3(x, N)| \leq \epsilon \quad \text{whenever } x \in \mathbb{R} \text{ and } N \geq O.$$

Hint: Use (2).

Exercise 1.7. Combine all these estimates we see that (3) holds.

Theorem 1.2. Suppose $f, g \in \mathcal{D}(\mathbb{R})$. We have

$$\int f(x)\overline{g(x)}\, dx = \frac{1}{2\pi} \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

Proof. Using the Fourier Inversion Formula we calculate

$$\begin{aligned}
2\pi \int f(x)\overline{g(x)}\, dx &= \int f(x) \overline{\left(\int e^{ix\xi} \hat{g}(\xi)\, d\xi \right)}\, dx \\
&= \int f(x) \left(\int e^{-ix\xi} \overline{\hat{g}(\xi)}\, d\xi \right) dx \\
&= \int \left(\int e^{-ix\xi} f(x)\, dx \right) \overline{\hat{g}(\xi)}\, d\xi \\
&= \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi.
\end{aligned}$$

□

Corollary 1.3. (Plancherel's Theorem.) Suppose $f \in \mathcal{D}(\mathbb{R})$. Then

$$\int |f(x)|^2\, dx = \frac{1}{2\pi} \int |\hat{f}(\xi)|^2, d\xi.$$

Remark 1.1. The preceding result extends to square summable functions by approximation.

1.2. **The Gaussian.** Let

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{for } x \in \mathbb{R}.$$

Then

$$\begin{aligned}
\sqrt{2\pi} \left(\frac{d}{d\xi} + \xi \right) \hat{g}(\xi) &= \int e^{-ix\xi} e^{-x^2/2} (-ix + \xi)\, dx \\
&= -i \int e^{-(ix\xi + x^2/2)}\, d_x \left(ix\xi + x^2/2 \right) \\
&= 0.
\end{aligned}$$

Exercise 1.8. Justify the steps in the preceding calculation.

Thus

$$\frac{d}{d\xi}\hat{g}(\xi) = -\xi\hat{g}(\xi).$$

It follows that

$$\hat{g}(\xi) = \hat{g}(0)e^{-\xi^2/2} = \sqrt{2\pi}\hat{g}(0)g(\xi).$$

Exercise 1.9. Justify the preceding assertion. It follows from uniqueness theorems from ordinary differential equations but you can and should do it directly using material we have already developed.

But

$$\int |g(x)|^2\,dx = \frac{1}{2\pi}\int |\hat{g}(\xi)|^2\,d\xi = |\hat{g}(0)|^2 \int |g(\xi)|^2\,d\xi$$

from which it follows that

$$1 = \hat{g}(0) = \int g(x)$$

so that

$$\hat{g} = \sqrt{2\pi}g.$$

1.3. **Another approach.** You may find this a lot easier. (So why didn't he do this first? Because he wanted to emphasize the link with Fourier series.)

Suppose $f, g \in \mathcal{D}(\mathbb{R})$.

Exercise 1.10. Show that

$$\int f(x)\hat{g}(x)\,dx = \int \hat{f}(\xi)g(\xi)\,d\xi.$$

Exercise 1.11. Use the preceding Exercise to show that

$$\int f(\epsilon x)\hat{g}(x) = \int \hat{f}(\eta)g(\epsilon\eta)\,d\eta \quad \text{for } \epsilon > 0.$$

Exercise 1.12. Let $\epsilon \downarrow 0$ in the preceding Exercise to obtain

$$f(0)\int \hat{g}(x)\,dx = g(0)\int \hat{f}(\eta)\,d\eta.$$

Exercise 1.13. Use an approximation argument to show that the result of the previous Exercise holds if $f, g \in \mathbf{Leb}_1$ and f, g are continuous at 0.

Exercise 1.14. Let $g(x) = e^{-|x|}$ for $x \in \mathbb{R}$. Calculate

$$\hat{g}(\xi) = \frac{2}{1+\xi^2} \quad \text{for } \xi \in \mathbb{R}.$$

Voilà! We have another proof of the Fourier inversion formula.