Final Problem Set.

1. Getting caught up.

The following exercises from previous problem sets I consider important. Do them before you do any others if you are trying to get caught up.

(i) Homework One: Exercise 2.1;
(ii) Homework Three: 3;
(iii) Homework Six: 2.1, 2.2, 4.1, 4.2;
(iv) Homework Seven: Oscillation, 3.1, 3.2;
(v) Homework Eight: 1.1, 1.2, 1.5;
(vi) Homework Nine: all of it.

2. Some topology.

Suppose X and Y are topological spaces and $f : X \to Y$.

1. Suppose $\mathcal{U}$ is a family of open subsets of X; $X| \cup \mathcal{U}$; and $f|U$ is continuous on U for each $U \in \mathcal{U}$. Show that f is continuous.

2. Suppose $\mathcal{F}$ is a finite family of closed subsets of X; $X| \cup \mathcal{F}$; and $f|F$ is continuous on F for each $F \in \mathcal{F}$. Show that f is continuous.

3. Show by counterexample that if $\mathcal{F}$ is not finite in 2. then f need not be continuous.

3. Power series.

Do **Exercise 1.4** in the notes on **Power Series**.

4. Leibniz' rule.

1. State and prove a version of Leibniz' Rule. You may make use of anything in **Differentiation on the line** that might help.

2. Carefully state and prove a version of integration by parts.

5. Change of variables in integrals.

Suppose

(i) $a, b, c, d \in \mathbb{R}$, $a < b$, c, d;
(ii) $g : (a, b) \to (c, d)$ is continuously differentiable, $g'(x) > 0$ for $a < x < b$, $\lim_{x \downarrow a} g(x) = c$ and $\lim_{x \uparrow b} g(x) = d$;
(iii) f is either Riemann(Lebesgue) integrable on (c, d).

Show that $(a, b) \ni x \mapsto f(g(x))g'(x)$ is Riemann(Lebesgue) integrable and that

$$\int_c^d f(y)\, dy = \int_a^b f(g(x))g'(x)\, dx.$$

Hint: Reduce it to the case when f is constant and use the appropriate version of the Fundamental Theorem of Calculus which occurred in a previous Homework Set. It is up to you to identity the correct version.

6. A compactness theorem.

Suppose B is a bounded subset of $\mathbb{R}^n$, $0 \leq M < \infty$ and X is the family of functions $f : B \to \mathbb{R}$ such that

$$||f|| = \sup\{|f(x)| : x \in B\} \leq M$$

and $\mathbf{Lip}\,(f) \leq M$. Show that X is complete and totally bounded with respect to the metric $X \times X \ni (f, g) \mapsto ||f - g||$. (The completeness follows pretty much from stuff we did with uniform convergence.)

7. The Weierstrass function.

Let

$$w : \mathbb{R} \to [0, 1]$$

be such that $w(x) = 1 - |x|$ for $|x| \leq 1$ and w is 2-periodic. Let

$$W(x) = \sum_{n=0}^{\infty} 2^{-n} w(2^n x) \quad \text{for } x \in \mathbb{R}.$$

1. Show that W is continuous. (Hint: Use the stuff on uniform convergence if you like, although it's not hard to do directly.

2. (Difficult!) Show that W is differentiable nowhere.

8. Fourier series.

Do **Exercises 1.1-1.4** from Fourier series. These are fairly routine.

9. Fourier integrals.

Do the **Exercises from the Fourier transform on the line.** The most difficult are **1.4, 1.5** and **1.6**. You could put these off till last; they are somewhat challenging but well within the capabilities of many of you and are well worth doing.