The Contraction Mapping Principle. Suppose (X, ρ) is a complete metric space,

$$C : X \to X \quad \text{and} \quad \mathbf{Lip}\,(C) < 1.$$

Then there is a unique point $a \in X$ such that $C(a) = a$. Moreover, for any $x \in X$ and any nonnegative integer n we have

$$\rho(a, C^n(x)) \leq \frac{\mathbf{Lip}\,(C)^n}{1 - \mathbf{Lip}\,(C)} \rho(C(x), x).$$

Proof. Suppose $x \in X$. By induction on n we infer that

$$\rho(C^{n+1}(x), C^n(x)) \leq \mathbf{Lip}\,(C)^n \rho(C(x), x)$$

for any nonnegative integer n. Using this inequality and the triangle inequality for ρ we infer that

$$(1) \quad \rho(C^m(x), C^n(x)) \leq \sum_{i=n}^{m-1} \rho(C^{i+1}(x), C^i(x)) \leq \sum_{i=n}^{m-1} \mathbf{Lip}\,(C)^i \rho(C(x), x) \leq \frac{\mathbf{Lip}\,(C)^n}{1 - \mathbf{Lip}\,(C)} \rho(C(x), x)$$

for any nonnegative integers with $m \geq n$. In particular, $\mathbf{N} \ni n \mapsto C^n(x)$ is a Cauchy sequence and therefore converges to some limit $a \in X$ as $n \uparrow \infty$. We have

$$C(a) = \lim_{n \to \infty} C^n(x) = C(\lim_{n \to \infty} C^n(x)) = \lim_{n \to \infty} C^{n+1}(x) = a$$

so a is a fixed point of C, as desired. Letting $m \uparrow \infty$ in (1) gives the asserted estimate for the distance of a to $C^n(x)$.

Finally, if $b \in X$ and $C(b) = b$ the

$$\rho(a, b) = \rho(C(a), C(b)) \leq \mathbf{Lip}\,(C)\, \rho(a, b) < \rho(a, b)$$

which implies that $\rho(a, b) = 0$ so $a = b$. □