The Contraction Mapping Principle. Suppose \((X, \rho)\) is a complete metric space, 
\[ C : X \to X \quad \text{and} \quad \text{Lip}(C) < 1. \]

Then there is a unique point \(a \in X\) such that \(C(a) = a\). Moreover, for any \(x \in X\) and any nonnegative integer \(n\) we have
\[ \rho(a, C^n(x)) \leq \frac{\text{Lip}(C)^n}{1 - \text{Lip}(C)} \rho(C(x), x). \]

**Proof.** Suppose \(x \in X\). By induction on \(n\) we infer that
\[ \rho(C^{n+1}(x), C^n(x)) \leq \text{Lip}(C)^n \rho(C(x), x) \]
for any nonnegative integer \(n\). Using this inequality and the triangle inequality for \(\rho\) we infer that
\[ \rho(C^m(x), C^n(x)) \leq \sum_{i=n}^{m-1} \rho(C^{i+1}(x), C^i(x)) \leq \sum_{i=n}^{m-1} \text{Lip}(C)^i \rho(C(x), x) \leq \frac{\text{Lip}(C)^n}{1 - \text{Lip}(C)} \rho(C(x), x) \]
for any nonnegative integers with \(m \geq n\). In particular, \(\mathbb{N} \ni n \mapsto C^n(x)\) is a Cauchy sequence and therefore converges to some limit \(a \in X\) as \(n \uparrow \infty\). We have
\[ C(a) = \lim_{n \to \infty} C^n(x) = C(\lim_{n \to \infty} C^n(x)) = \lim_{n \to \infty} C^{n+1}(x) = a \]
so \(a\) is a fixed point of \(C\), as desired. Letting \(m \uparrow \infty\) in (1) gives the asserted estimate for the distance of \(a\) to \(C^n(x)\).

Finally, if \(b \in X\) and \(C(b) = b\) the
\[ \rho(a, b) = \rho(C(a), C(b)) \leq \text{Lip}(C) \rho(a, b) < \rho(a, b) \]
which implies that \(\rho(a, b) = 0\) so \(a = b\). \(\Box\)