The coarea formula.

Suppose U is an open subset of $\mathbf{R}^n$ and

$$f : U \to \mathbf{R}$$

is a continuously differentiable function such that

$$\nabla f(\mathbf{x}) \neq \mathbf{0} \quad \text{for all } \mathbf{x} \in U.$$

Using the theory we have developed we know that

$$M_y = \{\mathbf{x} \in U : f(\mathbf{x}) = y\} \in \mathbf{M}_{n-1,n} \quad \text{for all } y \in \mathbf{R}.$$

There is a continuous function

$$h : U \to (0, \infty)$$

with the property that

$$\int_U g(\mathbf{x})h(\mathbf{x})\, d\mathbf{x} = \int_0^\infty \Big(\int g(\mathbf{w})\, d\|M_y\|\mathbf{w} \Big)\, dy \tag{1}$$

for each Borel function

$$g : U \to [0, \infty].$$

Your job is to determine what h is and prove (1). Start out assumuming f is linear. You will need to use the Change of Variables Formula for Multiple Integrals.

After you've done this, try to formulate and prove the analagous formula in the case when f takes values in $\mathbf{R}^l$ where $1 < l < n$.

I'll tell you what the answer is a very useful situation. Suppose $U = \mathbf{R}^n \sim \{\mathbf{0}\}$ and $f(\mathbf{x}) = |\mathbf{x}|$ for $\mathbf{x} \in U$. Then

$$\int_{\mathbf{R}^n \sim \{\mathbf{0}\}} g(\mathbf{x})\, d\mathbf{x} = \int_0^\infty \Big(\int g(\mathbf{w})\, d\|\{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| = r\}\|\mathbf{w} \Big)\, dr \tag{2}$$

for each Borel function $g : U \to [0, \infty]$ which is to say that h is identically 1. Incidentally, the integral on the right hand side of (2) equals

$$\int_0^\infty \Big(\int g(r\mathbf{u})\, d\|\mathbf{S}^{n-1}\|\mathbf{u} \Big) r^{n-1}\, dr;$$

prove this.