1. Initial segments, well ordering and the axiom of choice.

1.1. **Initial segments.** We suppose throughout this subsection that $<$ linearly orders the set X.

Definition 1.1. A subset I of X is an **initial segment** if

$$x \in X,\ y \in I \text{ and } x < y \ \Rightarrow\ x \in I.$$

Trivially, $\emptyset$ and X are initial segments.

Proposition 1.1. The union of a family of initial segments is an initial segment. The intersection of a nonempty family of initial segments is an initial segment.

Exercise 1.1. Prove this Proposition. If you understand the union and intersection of families of sets its simple.

Proposition 1.2. Suppose I and J are initial segments. Then either $I \subset J$ or $J \subset I$.

Proof. It will suffice to show that

$$J \sim I \neq \emptyset \ \Rightarrow\ I \subset J. \tag{1}$$

So suppose $y \in J \sim I$ and, contrary to (1), there is $x \in I \sim J$. Since $x \neq y$ we have *either* (i) $x < y$ *or* $y < x$ by trichotomy. If $x < y$ we have $x \in J$ since J is an initial segment and if $y < x$ we have $y \in I$ since I is an initial segment, neither of which is possible since $x \notin J$ and $y \notin I$. Thus (1) holds. $\square$

Remark 1.1. let $\mathcal{I}$ be the family of initial segments. For $I, J \in \mathcal{I}$ declare

$$I \prec J$$

if $I \subset J$ and $I \neq J$.

From Proposition 1.2 we see that $\prec$ is linear.

Now suppose $\mathcal{A}$ is a nonempty subfamily of $\mathcal{I}$. By Proposition 1.1 we find that $J = \cup\mathcal{A} \in \mathcal{I}$. It is easy to see that J is a least upper bound for $\mathcal{A}$ with respect to $\prec$.

We will construct the real numbers $\mathbb{R}$ from the rational numbers $\mathbb{Q}$ by using a slight variant of this construction.

Definition 1.2. For each $x \in X$ we let

$$\mathbb{I}(x) = \{w \in X : w < x\}.$$

Proposition 1.3. $\mathbb{I}(x)$ is an initial segment for any $x \in X$.

Proof. This follows directly from the transitivity of $<$. $\square$

Theorem 1.1. The linear ordering $<$ is complete if and only if the only initial segments are $\emptyset$, X and the sets

$$\mathbb{I}(x) \text{ and } \mathbb{I}(x) \cup \{x\}, \quad x \in X.$$

Exercise 1.2. Prove this Theorem.

Exercise 1.3. Let $<$ be the standard linear ordering on the set $\mathbb{Q}$ of rational numbers. (We will construct $\mathbb{Q}$ from the natural numbers $\mathbb{N}$ shortly.) Let

$$I = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}.$$

Show that I is an initial segment. (This is easy.) Show that

$$q \in \mathbb{Q} \text{ and } I = \mathbb{I}(q) \ \Rightarrow q^2 = 2.$$

(This gives many students fits.) We will show shortly that $q^2 = 2$ for no $q \in \mathbb{Q}$.

1.2. **Well ordering.** We suppose throughout this subsection that $<$ well orders the nonempty set X.

For each nonempty subset A of X we let

$$\mathbf{l}(A)$$

be the unique least member of A and we let

$$0 = \mathbf{l}(X).$$

Proposition 1.4. Suppose I is an initial segment of X, $I \neq X$ and

$$x = \mathbf{l}(X \sim I).$$

Then

$$I = \mathbb{I}(x).$$

Proof. Suppose $w \in \mathbb{I}(x)$. Were it the case that $w \in X \sim I$ we would have $x \leq w$ since x the least member of $X \sim I$; since $w < x$ this is excluded by trichotomy. So $\mathbb{I}(x) \subset I$.

Suppose $w \in I$. Since $x \in X \sim I$ we have $w \neq x$ so, by trichotomy, *either* (i) $w < x$ *or* (ii) $x < w$. Were it the case that $x < w$ we would have $x \in I$ since I is an initial segment. Thus (i) holds. So $I \subset \mathbb{I}(x)$. □

Definition 1.3. Let

$$X' = \{x \in X : x < y \text{ for some } y \in X\}.$$

(So $X' = X$ if X has no largest member and $X' = X \sim \{b\}$ if X has a largest member b.)

Let

$$S : X' \to X$$

be such that

$$S(x) = \mathbf{l}(\{y \in X : x < y\}) \quad \text{whenever } x \in X'.$$

We call S the **successor function**.

An element of $X \sim (\{0\} \cup \mathbf{rng}\, S)$ is called a **limit point**.

Definition 1.4. We say a subset A of X is **inductive** if $0 \in A$ and

$$\mathbb{I}(x) \subset A \ \Rightarrow \ x \in A \quad \text{whenever } x \in X.$$

Theorem 1.2. The principle of transfinite induction. Suppose A is and inductive subset of X. Then $A = X$.

Proof. Suppose, to the contrary, that $X \sim A$ is nonempty and let $x = \mathbf{l}(X \sim A)$. Then $\mathbb{I}(x) \subset A$ by trichotomy which implies $x \in A$ since A inductive. This is a contradiction. □

Theorem 1.3. Defining a Function by Transfinite Induction. Suppose

(i) Y is a set;
(ii) $\mathcal{G} = \{g : \text{for some } x, x \in X \text{ and } g : \mathbb{I}(x) \to Y\}$;
(iii) $G : \mathcal{G} \to Y$.

Then there is one any only one f such

$$f : X \to Y$$

and such that

$$f(x) = G(f|\mathbb{I}(x)) \quad \text{whenever } x \text{ in } X.$$

Proof. Let $\mathcal{H}$ be the family of functions h mapping initial segments J into X such that

$$h(x) = G(h|\mathbb{I}(x)) \quad \text{whenever } x \in J.$$

Let $f = \cup\mathcal{H}$.

Lemma 1.1. If $h_1, h_2 \in \mathcal{H}$ then either $h_1 \subset h_2$ or $h_2 \subset h_1$.

Proof. Let J_1, J_2 be the domains of h_1, h_2, respectively and note that, by a previous Proposition, either $J_1 \subset J_2$ or $J_2 \subset J_1$. Suppose $J_1 \subset J_2$ and let $K = \{x \in J_1 : h_1(x) \neq h_2(x)\}$. Suppose $K \neq \emptyset$ and let x be its least member. Then $h_1|\mathbb{I}(x) = h_2|\mathbb{I}(x)$ so $h_1(x) = G(h_1|\mathbb{I}(x)) = G(h_2|\mathbb{I}(x)) = h_2(x)$ so $x \in K$, a contradiction. Thus $h_1 \subset h_2$. In a similar fashion one shows that if $J_2 \subset J_1$ then $h_2 \subset h_1$. □

Thus f is a function. It is a simple matter to show that its domain is inductive as is the set of $x \in X$ such that $f(x) = G(f|\mathbb{I}(x))$. □

Theorem 1.4. Suppose X_i is well ordered by $<_i$, $i = 1, 2$. There is one and only one function

$$f$$

such that

(i) *either* the domain of f equals X_1 and the range of f is an initial segment of X_2 *or* the domain of f is an initial segment of X_1 and the range of f is an initial segment of X_2;
(ii) if $x, y \in \mathbf{dmn}\, f$ and $x <_1 y$ then $f(x) <_2 f(y)$.

Remark 1.2. Both (i) and (ii) may hold in which case we would say that the orderings $<_1$ and $<_2$ are **isomorphic**. In any case, the Theorem says that, given two well orderings, one may be thought of as the restriction of the other to an initial segment. In particular, any two well orderings are comparable.

Proof. Let $\mathcal{G}$ be the family of functions g mapping some initial segment I_1 of X_1 into X_2 such that

$$g(x) = \mathbf{l}(g[\mathbb{I}(x)]) \quad \text{whenever } x \in I_1$$

and let $f = \cup\mathcal{G}$.

Proceed as in the proof of the preceding Theorem. □

1.3. **Ordinal numbers.** It is clear what it means for two well ordered sets to be isomorphic and that we may define an equivalence relation on the family of all well ordered sets declaring two well ordered sets to be equivalent if they are isomorphic in the same way we defined cardinal numbers. The resulting equivalence classes are called **ordinal numbers**. The preceding Theorem implies that *there is a natural well ordering on the set (?!) of ordinal numbers.*

1.4. **Choice functions and the axiom of choice.**

Definition 1.5. Suppose $\mathcal{C}$ is a family of nonempty sets. We say c is a **choice function for** $\mathcal{C}$ if c is a function, $\mathbf{dmn}\, c = \mathcal{C}$ and

$$c(C) \in C \quad \text{whenever } C \in \mathcal{C}.$$

The Axiom of Choice (AC). Suppose $\mathcal{C}$ is a family of nonempty sets. Then there is a choice function for $\mathcal{C}$.

The Well Ordering Axiom (WO). Every set can be well ordered.

Theorem 1.5. (AC)$\Leftrightarrow$(WO).

Proof. Suppose (AC). Let X be a set and let ξ be a choice function on $2^X \sim \{\emptyset\}$. We will show that there is one and only one well ordering of X such that for any initial segment J of X not equal to X the least element of $X \sim J$ is $\xi(X \sim J)$. We prove this as follows. We let $\mathcal{W}$ be the set of ordered pairs(w, W) such that W is a subset of X, w is a well ordering of W and such that if J is an initial segment of W relative to w which is not equal to W then $\xi(X \sim J)$ is the least element with respect to w of $W \sim J$. Then

$$\bigcup\{w : \text{ for some } W,\ (w, W) \in \mathcal{W}\}.$$

is the desired well ordering of X. We leave the details to the reader; use the ideas in our previous results on well ordering.

Now suppose (WO) and let $\mathcal{C}$ be a family of nonempty sets. Let $X = \{(C, c) : C \in \mathcal{C} \text{ and } c \in C\}$ and let w be a well ordering of X. Then

$$\{(C, c) : C \in \mathcal{C} \text{ and } (C, c) \text{ is the } w\text{-least member of } \{(C, b) : b \in C\}\}$$

is a choice function for $\mathcal{C}$. □

Henceforth we go with the crowd and assume (AC).