The change of variables formula for multiple integrals.

Let \( n \) be a positive integer.

**Theorem.** Suppose \( U \) is an open subset of \( \mathbb{R}^n \),

\[
 f : U \to \mathbb{R}^n
\]

and the following conditions hold:

(i) \( f \) is continuously differentiable;

(ii) \( f \) is univalent and

(iii) \( \ker \partial f(a) = \{0\} \) whenever \( a \in U \).

Then

\[
 \mathcal{L}^n(f[A]) = \int_A |\det \partial f(x)| \, dx
\]

whenever \( A \) is a Lebesgue measurable subset of \( U \).

**Proof.** We set

\[
 ||| x ||| = \max\{|x_i| : i = 1, \ldots, n\} \quad \text{for each} \quad x \in \mathbb{R}^n
\]

and note that \( ||| \cdot ||| \) is a norm on \( \mathbb{R}^n \). We let \( ||| \cdot ||| \) be the corresponding norm on \( \mathbf{L}(\mathbb{R}^n; \mathbb{R}^n) \); that is, for each \( l \in \mathbf{L}(\mathbb{R}^n, \mathbb{R}^n) \) we set

\[
 ||| l ||| = \sup\{|l(v)|| : v \in \mathbb{R}^n \text{ and } |||v|| = 1\}.
\]

For each compact cube \( C \) such that \( C \subset U \) we let \( a(C) \) be the center of \( S \); we let \( R(C) \) be the halfsidelength of \( C \); we note that

\[
 C = \{x \in \mathbb{R}^n : ||x - a|| \leq R(C)\};
\]

we let

\[
 \alpha(C) = \sup\{||| \partial f(x) - \partial f(a(C)) ||| : x \in C\};
\]

and we let

\[
 \beta(C) = \inf\{|||\partial f(a(C))(u)||| : u \in \mathbb{R}^n \text{ and } |||u|| = 1\}.
\]

Owing to simple approximation arguments we may assume that \( A \) is a compact cube. For each \( \delta > 0 \) let

\[
 A(\delta) = \sup\{||| \partial f(x) - \partial f(a) ||| : x, a \in A \text{ and } |||x - a||| \leq \delta\}
\]

. Since \( \partial f \) is continuous and \( A \) is compact we find that

\[
 \lim_{\delta \to 0} A(\delta) = 0.
\]

For each \( x \in U \) let \( b(x) = \inf\{|||\partial f(x)(u)||| : |||u|| = 1\} \). Since \( b(x) = |||\partial f(x)^{-1}|||^{-1} \) for each \( x \in U \) we find that \( b \) is a positive continuous function on \( U \). We set

\[
 \mathbf{B} = \inf\{b(x) : x \in A\}.
\]

Since \( b \) is continuous and \( A \) is compact we infer that \( \mathbf{B} > 0 \) so we may choose \( \delta_0 > 0 \) such that

\[
 A(\delta_0) < \mathbf{B}. \quad (3)
\]

Suppose \( 0 < \eta \leq \delta_0 \). We let \( \mathcal{C} \) be a family of compact cubes with the following properties:
\( A = \cup \mathcal{C} \);

(5) \( \| C \cap D \|_n = 0 \) whenever \( C, D \in \mathcal{C} \);

(6) the halflength of any side of any member of \( \mathcal{C} \) does not exceed \( \eta \).

Suppose \( C \in \mathcal{C} \). Because \( \alpha(C) \leq \mathbf{A}(\delta_0) < \mathbf{B} \leq \beta(C) \)
we may apply the Inverse Function Theorem with \( f, L, a, R \) there equal \( f|C, \partial f(a(C)), a(C), R(C) \), respectively, to conclude that
\[
\{ f(a(C)) + \partial f(a(C))(h) : (1 - \alpha(C) / \beta(C))||h|| \leq R(C) \}
\subset f[C]
\subset \{ f(a(C)) + \partial f(a(C))(h) : (1 + \alpha(C) / \beta(C))||h|| \leq R(C) \}.
\]

Combining this with our earlier results about how areas change under linear maps we infer that
\[
(1 - \alpha(C) / \beta(C))^n |\text{det} \partial f(a(C))| ||C||_n
= \mathcal{L}^n(\{ f(a(C)) + \partial f(a(C))(h) : (1 - \alpha(C) / \beta(C))||h|| \leq R(C) \})
\leq \mathcal{L}^n(f[C])
\leq \mathcal{L}^n(\{ f(a(C)) + \partial f(a(C))(h) : (1 + \alpha(C) / \beta(C))||h|| \leq R(C) \})
= (1 + \alpha(C) / \beta(C))^n |\text{det} \partial f(a(C))| ||C||_n.
\]

Setting
\[
I = \sum_{C \in \mathcal{C}} |\text{det} \partial f(a(C))| ||C||_n
\]
and keeping in mind (2) and (3) we find that
\[
(1 - \mathbf{A}(\eta) / \mathbf{B})^n I \leq \sum_{C \in \mathcal{C}} \mathcal{L}^n(f[C]) \leq (1 + \mathbf{A}(\eta) / \mathbf{B})^n I.
\]

Because \( f \) is univalent we find that
\[
\sum_{C \in \mathcal{C}} \mathcal{L}^n(f[C]) = \sum_{C \in \mathcal{C}} \mathcal{L}^n(f[\text{int} C]) = \mathcal{L}^n(f[\cup_{C \in \mathcal{C}} \text{int} C]) = \mathcal{L}^n(f[A]).
\]

Since \( I \) is a Riemann sum for \( J = \int_A |\text{det} \partial f(x)| \, dx \) with respect to cubes of halflength \( \eta \) and since (2) holds we infer that \( J = n(f[A]) \), as desired. \( \square \)