How elementary linear maps change areas.
Fix an integer \( n \geq 2 \). Let
\[
C = \{ x \in \mathbb{R}^n : 0 \leq x_i \leq 1, \ i = 1, \ldots, n \}.
\]
The “\( C \)” here stands for cube. Our goal in this Introduction is to prove that
\[
(1) \quad |L[C]| = |\det L|, \quad L \in \text{GL}(\mathbb{R}^n).
\]
For each \( c \in \mathbb{R} \sim \{0\} \) let \( S_c \in \text{GL}(\mathbb{R}^n) \) be defined by
\[
S_c(x) = (x_1, x_2, \ldots, x_n + cx_{n-1}), \quad x \in \mathbb{R}^n.
\]
The ‘\( S \)’ here stands for shear; that this is reasonable terminology can be seen by drawing a picture of what \( S_c \) does to the cube \( C \). For each \( c > 0 \) let \( D_c \in \text{GL}(\mathbb{R}^n) \) be defined by
\[
D_c(x) = (x_1, x_2, \ldots, cx_n), \quad x \in \mathbb{R}^n.
\]
The ‘\( D \)’ here stands for dilate. For each \( \sigma \in \mathbb{R}^n \) let \( P_\sigma \in \text{L}(\mathbb{R}^n, \mathbb{R}^n) \) be defined by
\[
P_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad x \in \mathbb{R}^n.
\]
The ‘\( P \)’ here stands for permutation.
It follows from the the Gaussian elimination algorithm from elementary linear algebra that any member of \( \text{GL}(\mathbb{R}^n) \) can be written as a product of shears, dilations and permutations. Thus, if we can show that (1) holds for shears, permutations and dilations we will, in view of the product rule for determinants, have shown that (1) holds for any \( L \in \text{GL}(\mathbb{R}^n) \).
Suppose \( c \in \mathbb{R} \sim \{0\} \). Then
\[
S_c[C] = \{ (y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n : 0 \leq y_1 \leq 1, \ldots, 0 \leq y_{n-1} \leq 1, cy_{n-1} \leq y_n \leq 1 + cy_{n-1} \}
\]
so
\[
|S_c[C]| = \int_0^1 \left( \int_{cy_{n-1}}^{1+cy_{n-1}} dy_n \right) dy_{n-1} = 1 = |\det S_c|.
\]
Also,
\[
D_c[C] = \{ (y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n : 0 \leq y_1 \leq 1, \ldots, 0 \leq y_{n-1} \leq 1, 0 \leq y_n \leq c \}
\]
so
\[
|D_c[C]| = |c| = |\det D_c|.
\]
Finally, \( P_\sigma[C] = C \) so
\[
|P_\sigma[C]| = |C| = 1 = |\det P_\sigma|.
\]