1. Alternating and symmetric multilinear functions.

Suppose V is a vector space.

Definition 1.1. For each vector space Z and each $p \in \mathbb{Z}$ we set

$$\bigotimes^p (V,Z) = \begin{cases} \{0\} & \text{if } p < 0, \\ Z & \text{if } p = 0; \\ \mathbb{Lin}(V,Z) & \text{if } p = 1, \\ \mathbb{MultiLin}(V^p, Z) & \text{if } p \geq 1. \end{cases}$$

If $\mu \in \bigotimes^p (V,Z)$ then μ is **symmetric** if $p \leq 1$ or $p > 1$ and

$$\mu(v \circ \sigma) = \mu(v) \text{ whenever } \sigma \text{ is a transposition of } [\![1,p]\!]$$

and μ is **alternating** or **antisymmetric** if $p \leq 1$ or $p > 1$ and

$$\mu(v \circ \sigma) = -\mu(v) \text{ whenever } \sigma \text{ is a transposition of } [\![1,p]\!].$$

We let

$$\bigodot^p (V,Z) = \{\mu \in \mathbb{MultiLin}(V^p, Z) : \mu \text{ is symmetric}\}$$

and we let

$$\bigwedge^p (V,Z) = \{\mu \in \mathbb{MultiLin}(V^p, Z) : \mu \text{ is alternating}\}.$$

Evidently, $\bigwedge^p (V,Z)$ and $\bigodot^p (V,Z)$ are linear subspaces of $\bigotimes^p (V,Z)$.

If U is a vector space and $l \in \mathbb{Lin}(U,V)$ we define the linear map

$$\bigotimes^p (l,Z) : \bigotimes^p (V,Z) \to \bigotimes^p (U,Z)$$

by setting

$$\bigotimes^p (l,Z)(\varphi)(u) = \varphi(v)$$

for $\varphi \in \bigotimes^p (V,Z)$, $u \in U^p$ and where $v \in V^p$ is such that $v_i = l(u_i)$ for $i \in [\![1,p]\!]$. This extends the notion of adjoint encountered previously. We note that $\bigotimes^p (l,Z)$ preserves symmetry and antisymmetry and we set

$$\bigodot^p (l,Z) = \bigotimes^p (l,Z) \Big| \bigodot^p (V,Z) \quad \text{and} \quad \bigwedge^p (l,Z) = \bigotimes^p (l,Z) \Big| \bigwedge^p (V,Z).$$

One easily verifies that if W is a vector space and $m \in \mathbb{Lin}(V,W)$ then

$$\bigotimes^p (m \circ l, Z) = \bigotimes^p (l,Z) \circ \bigotimes^p (m,Z)$$

and that similar formulae hold with $\bigotimes^p (\cdot, Z)$ replaced by $\bigodot^p (\cdot, Z)$ and $\bigwedge^p (\cdot, Z)$.

1.1. **Bases.** Suppose E is a basis for V and $p \in \mathbb{N}^+$. For each $e \in E^p$ we let

$$e^* \in \bigotimes^p V$$

be such that

$$e^*(v) = \Pi_{i=1}^p e_i^*(v_i) \quad \text{for } v \in V^p.$$

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Interior multiplication. For each $p \in \mathbb{Z}$ we define the bilinear map

$$\bigotimes^p (V,Z) \times V \xrightarrow{\llcorner} \bigotimes^{p-1} (V,Z)$$

as follows: Given $\varphi \in \bigotimes^p(V,Z)$ and $v \in V$ we set $\varphi \llcorner v = 0$ in case $p \leq 0$, we set $\varphi \llcorner v = \varphi(v)$ in case $p = 1$ and, in case $p > 1$, we set

$$(\varphi \llcorner v)(w) = \varphi(\overline{v\, w}) \quad \text{for } w \in V^{p-1}.$$

We call $\varphi \llcorner v$ **interior multiplication** or **contraction of** φ **by** v. Note that interior multiplication by v preserves the subspaces of symmetric and alternating multilinear functions. For each $v \in V$ we define

$$\iota_v \in \mathbb{L}\mathrm{in}(\bigotimes^p(V,Z), \bigotimes^{p-1}(V,Z))$$

by letting $\iota_v(\varphi) = \varphi \llcorner v$ for $\varphi \in \bigotimes^p V$.

Definition 1.2. For each integer p we let

$$\bigotimes^p V = \bigotimes^p (V, \mathbf{R}), \quad \bigodot^p V = \bigodot^p (V, \mathbf{R}), \quad \bigwedge^p V = \bigwedge^p (V, \mathbf{R})$$

and we let

$$\bigotimes^p l = \bigotimes^p (l, \mathbf{R}), \quad \bigodot^p l = \bigodot^p (l, \mathbf{R}), \quad \bigwedge^p l = \bigwedge^p (l, \mathbf{R}).$$

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Theorem 1.1 (The Contravariant Exterior Product.). There is one and only one map

$$\bigwedge^p V \times \bigwedge^q V \xrightarrow{\wedge} \bigwedge^{p+q} V$$

such that, if $\varphi \in \bigwedge^p V$ and $\psi \in \bigwedge^q V$ then

(CE1) $\varphi \wedge \psi = \varphi \psi$ if $p = 0$ and $q = 0$;
(CE2) $(\varphi \wedge \psi) \llcorner v = (\varphi \llcorner v) \wedge \psi + (-1)^p \varphi \wedge (\psi \llcorner v)$ for all v in V.

This mapping is bilinear.

Remark 1.1. Because (CE2) holds we say $\varphi \mapsto \varphi \llcorner v$ is a **skewderivation**.

Proof. The statement holds trivially if $p < 0$ or $q < 0$ so suppose $p \geq 0$ and $q \geq 0$ and induct on $r = p + q$. It is evident by induction on r that there is unique map

$$\bigwedge^p V \times \bigwedge^q V \xrightarrow{\wedge} \bigotimes^{p+q} V$$

such that (CE1) and (CE2) are satisfied and that this map is bilinear. We need to show that if $\varphi \in \bigwedge^p V$ and $\psi \in \bigwedge^q V$ then $\varphi \wedge \psi$ is alternating. This is trivially the case if $r = 0$ so assume $r > 0$ and that the Theorem holds for smaller r.

Since

$$(\varphi \wedge \psi) \llcorner v = (\varphi \llcorner v) \wedge \psi + (-1)^p \varphi \wedge (\psi \llcorner v)$$

for any $v \in V$ the inductive hypothesis implies that $\varphi \wedge \psi$ is alternating in its last $r-1$ arguments. To complete the proof it will suffice to show that it is alternating in its first two arguments. That is, given $v, w \in V$ we need to show that $((\varphi \wedge \psi) \llcorner v) \llcorner w$ is alternating in v, w. But

$$\iota_w(\iota_v((\varphi \wedge \psi)) = \iota_w(\iota_v(\varphi) \wedge \psi + (-1)^p \varphi \wedge \iota_v(\psi)) = A + B + C + D$$

where

$$A = \iota_w(\iota_v(\varphi)) \wedge \psi \quad \text{and} \quad B = (-1)^{p-1} \iota_v(\varphi) \wedge \iota_w(\psi)$$
$$C = (-1)^p \iota_w(\varphi) \wedge \iota_v(\psi) \quad \text{and} \quad D = (-1)^p (-1)^p \varphi \wedge \iota_w(\iota_v(\psi)$$

A is alternating in v and w because φ is alternating; $B+C$ is clearly alternating in v and w; and D is alternating in v and w because ψ is alternating.

$$\begin{aligned}((\varphi\wedge\psi)\llcorner v)\llcorner w &= \big((\varphi\llcorner v)\wedge\psi+(-1)^p\varphi\wedge(\psi\llcorner v)\big)\llcorner w\\ &= \big((\varphi\llcorner v)\llcorner w\big)\wedge\psi+(-1)^{p-1}(\varphi\llcorner v)\wedge(\psi\llcorner w)\\ &+(-1)^p(\varphi\llcorner w)\wedge(\psi\llcorner v)+(-1)^p(-1)^p\varphi\wedge\big((\psi\llcorner v)\llcorner w\big).\end{aligned}$$

The sum of the second and third terms in this sum is clearly alternating in v and w and the first and fourth terms are alternating in v and w because φ and ψ are alternating. □

Theorem 1.2. Suppose $\varphi\in\bigwedge^p V$, $\psi\in\bigwedge^q V$ and $\zeta\in\bigwedge^r V$. Then

$$(\varphi\wedge\psi)\wedge\zeta=\varphi\wedge(\psi\wedge\zeta).$$

(That is, exterior multiplication is **associative**.)

Proof. The Theorem holds trivially if any of p,q,r are negative. So we assume that p,q,r are nonnegative and induct on $s=p+q+r$. The Theorem holds trivially if $s=0$ so suppose $s>0$ and that Theorem holds for smaller s. Given $v\in V$ we calculate

$$\begin{aligned}\big((\varphi\wedge\psi)\wedge\zeta\big)\llcorner v &= \big((\varphi\wedge\psi)\llcorner v\big)\wedge\zeta+(-1)^{p+q}(\varphi\wedge\psi)\wedge(\zeta\llcorner v)\\ &= \big((\varphi\llcorner v)\wedge\psi\big)\wedge\zeta+(-1)^p\big(\varphi\wedge(\psi\llcorner v)\big)\wedge\zeta\\ &\quad+(-1)^{p+q}(\varphi\wedge\psi)\wedge(\zeta\llcorner v);\end{aligned}$$

$$\begin{aligned}\big(\varphi\wedge(\psi\wedge\zeta)\big)\llcorner v &= (\varphi\llcorner v)\wedge(\psi\wedge\zeta)+(-1)^p\varphi\wedge\big((\psi\wedge\zeta)\llcorner v\big)\\ &= (\varphi\llcorner v)\wedge(\psi\wedge\zeta)\\ &\quad+(-1)^p\varphi\wedge\big((\psi\llcorner v)\wedge\zeta\big)+(-1)^p(-1)^q\varphi\wedge\big(\psi\wedge(\zeta\llcorner v)\big).\end{aligned}$$

Now apply the inductive hypothesis. □

Theorem 1.3. Suppose $\varphi\in\bigwedge^p V$ and $\psi\in\bigwedge^q V$. Then

$$\varphi\wedge\psi=(-1)^{pq}\psi\wedge\varphi.$$

(That is, exterior multiplication is **anticommutative** in the graded sense.)

Proof. The Theorem holds trivially if either p or 1 is negative. Induct on $r=p+q$. If $r=0$ this amounts to the commutative law for multiplication of real numbers so suppose $r>0$ and that the Theorem holds for smaller r. For any v in V we have

$$(\varphi\wedge\psi)\llcorner v=(\varphi\llcorner v)\wedge\psi+(-1)^p\varphi\wedge(\psi\llcorner v);$$

$$(-1)^{pq}(\psi\wedge\varphi)\llcorner v=(-1)^{pq}(\psi\llcorner v)\wedge\varphi+(-1)^{pq}(-1)^q\psi\wedge(\varphi\llcorner v).$$

Now apply the inductive hypothesis. □

Corollary 1.1. Suppose p is odd and $\varphi\in\bigwedge^p V$. Then

$$\varphi\wedge\varphi=0.$$

1.2. $\wedge^p$. For $p \in \mathbb{N}^+$ and $\omega \in (V^*)^p$ we define
$$\wedge^p(\omega) \in \bigwedge^p V$$
by setting $\wedge^1(\omega) = \omega_1$ and requiring that, if $p > 1$,
$$\wedge^p(\omega) = \omega_1 \wedge \wedge^{p-1}(\omega|[\![1, p-1]\!]).$$
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Suppose E is a basis for V. For each $a \in E$ we let $a^* \in V^*$ be such that
$$a^*(b) = \begin{cases} 1 & \text{if } b = a, \\ 0 & \text{if } b \in E \sim \{a\}. \end{cases}$$
Suppose $\prec$ is a well ordering of E. For each $p \in \mathbb{N}$ and each $A \in \Lambda(E,p)$ we let
$$\mathbf{e}_A \in E^p$$
be such that $\mathbf{rng}\,\mathbf{e}_A = A$ and $\mathbf{e}_A$ increasing with respect to $\prec$; we let
$$\mathbf{e}_A^* \in \{e^* : e \in E\}^p$$
be such that the i-th coordinate of $\mathbf{e}_A^*$ equals $(\mathbf{e}_A(i))^*$; and we let
$$\mathbf{e}^A = \wedge^p(\mathbf{e}_A^*) \in \bigwedge^p V.$$

Proposition 1.1. The following statements hold:

(i) if A is a finite subset of E and $b \in E \sim A$ then
$$\mathbf{e}^A \llcorner b = 0;$$
(ii) if A and B are finite subsets of E and $a \prec b$ whenever $a \in A$ and $b \in B$ then
$$\mathbf{e}^{A \cup B} = \mathbf{e}^A \wedge \mathbf{e}^B;$$
(iii) If A is a finite subset of E, $a \in A$, $B = \{x \in A : x \prec a\}$ and $C = \{x \in A : a \prec x\}$ then
$$\mathbf{e}^A \llcorner a = (-1)^{|B|}\mathbf{e}^{B \cup C};$$
(iv) if A and B are nonempty finite subsets of E then
$$\mathbf{e}^A(\mathbf{e}_B) = \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{if } A \neq B. \end{cases}$$
(iv) if $A \in \Lambda(E,p)$ and $e \in E^p$. Then
$$\mathbf{e}^A(e) = 0 \quad \text{if } \mathbf{rng}\, e \neq A$$
and, if $\mathbf{rng}\, e = A$ and $\sigma = \mathbf{e}_A^{-1} \circ e$, then $\sigma \in \Sigma(p)$ and
$$\mathbf{e}^A(e) = \mathbf{sgn}(\sigma).$$

Proof. If A is empty (i)-(iv) hold trivially. So suppose $A \neq \emptyset$. We prove (i)-(iii) by induction on $|A|$.

If $b \in E \sim A$ then, letting a be the $\prec$-first member of A and arguing inductively, we have
$$\mathbf{e}^A \llcorner b = (a^* \wedge \mathbf{e}^{A \sim \{a\}}) \llcorner b = a^*(b) \wedge \mathbf{e}^{A \sim \{a\}} - a^* \wedge ((\mathbf{e}^{A \sim \{a\}}) \llcorner b = 0 + 0 = 0$$
so (i) holds.

If A and B are as in (ii) then letting a be the $\prec$ first member of A and arguing inductively we find that

$$\begin{aligned}\mathbf{e}^{A\cup B} &= a^* \wedge \mathbf{e}^{(A\sim\{a\})\cup B}\\ &= a^* \wedge (\mathbf{e}^{A\sim\{a\}} \wedge \mathbf{e}^B)\\ &= (a^* \wedge \mathbf{e}^{A\sim\{a\}}) \wedge \mathbf{e}^B\\ &= \mathbf{e}^A \wedge \mathbf{e}^B\end{aligned}$$

so (ii) holds.

If A, a and B, C are as in (iii) we use (ii) and (i) and argue inductively to obtain

$$\begin{aligned}\mathbf{e}^A \llcorner a &= (\mathbf{e}^B \wedge a^* \wedge \mathbf{e}^C) \llcorner a\\ &= (\mathbf{e}^B \llcorner a) \wedge a^* \wedge \mathbf{e}^C + (-1)^{|B|}\mathbf{e}^B \wedge (a^* \llcorner a) \wedge \mathbf{e}^C\\ &\qquad + (-1)^{|B|+1}\mathbf{e}^B \wedge a^* \wedge (\mathbf{e}^C \llcorner a)\\ &= (-1)^{|B|}\mathbf{e}^B \wedge \mathbf{e}^C\\ &= (-1)^{|B|}\mathbf{e}^{B\cup C}\end{aligned}$$

so (iii) holds.

Suppose A is a nonempty finite subsets of E and $e \in V^{|A|}$. If $i \in [\![1, |A|]\!]$ and $e_i \notin A$ we let τ transpose 1 and i and let $f \in E^{|A|-1}$ be such that $e \circ \tau = \overline{e_i\, f}$. Then

$$\mathbf{e}^A(e) = -\mathbf{e}^A(e \circ \tau) = \mathbf{e}^A(\overline{e_i\, f}) = (\mathbf{e}^A \llcorner e_i)(f) = 0$$

by (i). If $\mathbf{rng}\, e = \mathbf{rng}\, A$ it evident that $\sigma = \mathbf{e}_A^{-1} \circ e \in \Sigma(|A|)$ so

$$\mathbf{e}_A(e) = \mathbf{e}_A(\mathbf{e}_A \circ \sigma) = \mathbf{sgn}(\sigma)\mathbf{e}^A(\mathbf{e}_A) = 1$$

since, letting $B = A \sim \{a\}$ and arguing inductively using (i),

$$\mathbf{e}_A(\mathbf{e}_A) = (a^* \wedge \mathbf{e}^B)(\overline{a\, \mathbf{e}_{A\sim\{A\}}}) = \left(a^*(a)\mathbf{e}^B - a^* \wedge ((\mathbf{e}^B) \llcorner a)\right)(\mathbf{e}_B = 1 + 0 = 1.$$

□

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Theorem 1.4. Suppose $\phi, \psi \in \bigwedge^p V$ and

$$\phi(\mathbf{e}_A) = \psi(\mathbf{e}_A) \quad \text{for all } A \in \Lambda(E, p).$$

Then $\phi = \psi$.

Proof. Suppose $e \in \Lambda(E, p)$ Let $A = \mathbf{rng}\, E$ and let $\sigma \in \Sigma(p)$ be such that $e = \mathbf{e}_A \circ \sigma$. Then

$$\phi(e) = \mathbf{sgn}(\sigma)\phi(\mathbf{e}_A) = \mathbf{sgn}(\sigma)\psi(\mathbf{e}_A) = \psi(e).$$

It follows from ?? that $\phi = \psi$. □

Corollary 1.2. Suppose $\phi \in \bigwedge^p V$. Then

$$\{v \in E^p : \phi(v) \neq 0\} \quad \text{is finite}$$

and

(1) $$\phi(v) = \sum_{A \in \Lambda(E,p)} \mathbf{e}^A(v)\phi(\mathbf{e}_A).$$

Proof. The first assertion of the corollary follows from ?? and that implies that the right hand sice of (1) defines a member of $\bigwedge^p(V,Z)$. That both sides of (1) have the same value on $\mathbf{e}_B$ for any $B \in \Lambda(E,p)$ follows from ??. □

Theorem 1.5. Suppose $\omega \in (V^*)^p$. Then

$$\wedge^p(\omega)(v) = \sum_{\sigma \in \Sigma(p)} \mathbf{sgn}(\sigma) \Pi_{i=1}^p \omega_i(v_{\sigma(i)}) \quad \text{for any } v \in V^p. \tag{2}$$

Proof. For each $v \in V^p$ let $\psi(v)$ be the right hand side of (2). So $\psi \in \bigotimes^p(V,Z)$, For $\rho \in \Sigma(p)$ and $v \in V^p$ we have

$$\begin{aligned}
\psi(v \circ \rho) &= \sum_{\sigma \in \Sigma(p)} \mathbf{sgn}(\sigma) \Pi_{i=1}^p \omega_i((v \circ \rho)_{\sigma(i)}) \\
&= \sum_{\sigma \in \Sigma(p)} \mathbf{sgn}(\sigma \circ \rho^{-1}) \Pi_{i=1}^p \omega_i(v \circ_{\sigma(i)}) \\
&= \mathbf{sgn}(\rho) \sum_{\sigma \in \Sigma(p)} \mathbf{sgn}(\sigma) \Pi_{i=1}^p \omega_i(v \circ_{\sigma(i)}) \\
&= \mathbf{sgn}(\rho)\psi(v).
\end{aligned}$$

Thus $\psi \in \bigwedge^p(V,Z)$. Since ?? implies that both sides of (2) have the same value on $\mathbf{e}_A$ for any $A \in \Lambda(E,p)$ we infer from ?? that (2) holds. □

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Corollary 1.3. Suppose $n = \mathbf{dim}\, V < \infty$. Then

$$\phi = \phi(\mathbf{e}_E)\mathbf{e}^E \quad \text{for } \phi \in \textstyle\bigwedge^n V. \tag{3}$$

Moreover, $\mathbf{e}^E(\mathbf{e}_E) = 1$ and $\{\mathbf{e}^E\}$ is basis for $\mathbf{dim} \bigwedge^n V$. In particular, $\mathbf{dim} \bigwedge^n V = 1$.

Proof. (3) holds since $\{A : A \subset E \text{ and } |A| = n\} = E$ by ?? which also implies that $\mathbf{span}\, \mathbf{e}_E = \bigwedge^n V$. That $\mathbf{e}^E(\mathbf{e}_E) = 1$ follows from Proposition ?? and this implies $\{\mathbf{e}^E\}$ is a basis for $\bigwedge^n V$. □

Proposition 1.2. For $\phi, \psi \in \bigwedge^n V$ with $\psi \neq 0$ there is unique

$$\frac{\phi}{\psi} \in \mathbb{R}$$

such that

$$\frac{\phi}{\psi} = \frac{\phi(v)}{\psi(v)} \quad \text{whenever } v \in V^n \text{ and } \mathbf{span}\,\mathbf{rng}\, v = V.$$

Moreover,

$$\phi = \frac{\phi}{\psi}\psi.$$

Proof. This is a straightforward consequence of the foregoing. □

Proposition 1.3. Suppose $L \in \mathbb{E}\mathrm{nd}(V)$. There is a unique $r \in \mathbb{R}$ such that

$$\left(\bigwedge^n L\right)(\phi) = r\xi \quad \text{for } \phi \in V^n. \tag{4}$$

Proof. This holds since $\bigwedge^n L \in \mathbb{E}\mathrm{nd}\left(\bigwedge^n V\right)$ and $\mathbf{dim} \bigwedge^n V = 1$. □

Definition 1.3. For $L \in \mathbb{End}(V)$ we let
$$\mathbf{det}\, L = r$$
where r is as in (4).

Theorem 1.6. Suppose $L, M \in \mathbb{End}(V)$. Then
$$\mathbf{det}\,(L \circ M) = (\mathbf{det}\, L)(\mathbf{det}\, M).$$

Proof. If $\phi \in \bigwedge^n V$ then
$$\begin{aligned}\mathbf{det}\,(L \circ M)\phi &= \left(\bigwedge^n (L \circ M)\right)(\phi)\\ &= \left(\bigwedge^n M\right)\left(\left(\bigwedge^n L\right)(\phi)\right)\\ &= \left(\bigwedge^n M\right)((\mathbf{det}\, L)\phi)\\ &= (\mathbf{det}\, L)\left(\left(\bigwedge^n M\right)(\phi)\right)\\ &= (\mathbf{det}\, L)(\mathbf{det}\, M)\phi.\end{aligned}$$
□

1.3. **The signature of a permutation revisited.** Suppose $n \in \mathbb{N}^+$. Let $\mathbf{e}_i$, $i \in [\![1, n]\!]$, be the standard basis vectors for $\mathbb{R}^n$. For $\rho \in \Sigma([\![1, n]\!])$ let $L_\rho \in \mathbb{GL}(\mathbb{R}^n)$ be such that
$$L_\rho(\mathbf{e}_i) = \mathbf{e}_{\rho(i)} \quad \text{for } i \in [\![1, n]\!].$$
A simple calculation shows that
$$L_\rho \circ L_\sigma = L_{\rho\circ\sigma} \quad \text{for } \sigma, \rho \in \Sigma([\![1, n]\!]).$$
It follows from ?? that
$$\Sigma([\![1, n]\!]) \ni \sigma \mapsto \mathbf{det}\, L_\sigma \in \{-1, 1\}$$
is a homomorphism. Since
$$\mathbf{det}\,\sigma = -1 \quad \text{if } \sigma \text{ is a transposition of } [\![1, n]\!]$$
we find that
$$\mathbf{sgn}(\sigma) = \mathbf{det}\, L_\sigma \quad \text{for } \sigma \in \Sigma([\![1, n]\!]).$$

Corollary 1.4. Suppose A and B are finite subsets of E, $|A| = |B|$ and σ is a permutation of $[\![1, |A|]\!]$. Then
$$\mathbf{e}^B(\mathbf{e}_A \circ \sigma) = \begin{cases}\mathbf{sgn}(\sigma) & \text{if } B = A,\\ 0 & \text{if } B \neq A.\end{cases}$$

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2. The shuffle formula.

Suppose $m, N \in \mathbb{N}^+$. We let
$$\mathcal{I}(m, N)$$
be the set of m-tuples I of finite nonempty subsets of $\mathbb{N}^+$ such that

(i) $I_i = [\![\min I, \max I]\!]$ for $i \in [\![1, m]\!]$;
(ii) $\min I_1 = 1$;
(iii) $\min I_{i+1} = \max I_i + 1$ for $i \in [\![1, m)\!)$;

(iv) $N = \sum_{i=1}^m |I_i|$.

If $I \in \mathcal{I}(m, N)$ we let

$$\mathbf{Sh}(I)$$

be the set of permutations σ of $[\![1, N]\!]$ such that $\sigma|I_i$ is increasing for $i \in [\![1, m]\!]$; such a σ is called a **shuffle of type** I. Evidently,

$$\mathbf{rev}(\sigma) = \bigcup_{i=1}^{m} \bigcup_{j=i+1}^{m} \{(k,l) \in I_i \times I_j : \sigma(i) > \sigma(j)\}. \tag{5}$$

Suppose $m \in \mathbb{N}^+$, p is an m-tuple of positive integers. Let $P_0 = 0$ and, for $i \in [\![1, m]\!]$, let $P_i = \sum_{j=1}^i p_i$. For $i \in [\![1, m]\!]$ we let $I_i = [\![P_{i-1}+1, P_i]\!]$; Thus $I \in \mathcal{I}(m)$.

Theorem 2.1. Suppose ϕ is an m-tuple such that $\phi_i \in \bigwedge^{p_i} V$ for $i \in [\![1, m]\!]$ and $v \in V^{P_m}$. Then

$$\left(\bigwedge_{i=1}^{m} \phi_i\right)(v) = \sum_{\sigma \in \mathbf{Sh}(I)} \mathbf{sgn}(\sigma) \Pi_{i=1}^m \phi_i(v \circ (\sigma|I_i)). \tag{6}$$

Proof. We prove this by induction on m. (6) holds trivially if $m = 1$. Suppose $w \in V^{P_m - 1}$ is such that $v = \overline{v_1\, w}$. □

2.1. **The case** $m = 2$**.** Let

$$\Omega = (\phi_1 \wedge \phi_2)(v); \quad \Omega_1 = ((\phi_1 \llcorner v_1) \wedge \phi_2)(w); \quad \Omega_2 = (-1)^{p_1}(\phi_1 \wedge (\phi_2 \llcorner v_1))(w);$$

Thus

$$\Omega = \Omega_1 + \Omega_2.$$

Lemma 2.1. We have

$$\Omega_1 = \sum_{\sigma \in \mathbf{Sh}(I),\ \sigma(1)=1} \mathbf{sgn}(\sigma) \phi_1(v \circ (\sigma|I_1)) \phi_2(v \circ (\sigma|I_2)). \tag{7}$$

Proof. Induct on p_1. If $p_1 = 1$ then $\Omega_1 = \phi_1(v_1)\phi_2(w)$ so (7) holds.

Suppose $p_1 > 1$. Let $J_1 = [\![1, p_1 - 1]\!]$ and let $J_2 = [\![p_1, p_1 + p_2 - 1]\!]$. Arguing inductively we find that

$$\begin{aligned}
\Omega_1 &= \sum_{\rho \in \mathbf{Sh}(J)} \mathbf{sgn}(\rho)(\phi_1 \llcorner v_1)(w \circ (\rho|J_1))\phi_2((w \circ (\rho|J_2)) \\
&= \sum_{\sigma \in \mathbf{Sh}(I), \sigma(1)=1} \mathbf{sgn}(\sigma)\phi_1(v \circ (\sigma|I_1))\phi_2((w \circ (\sigma|I_2)).
\end{aligned}$$

□

Lemma 2.2. We have

$$\Omega_2 = \sum_{\sigma \in \mathbf{Sh}(I),\ \sigma(1)=p_1+1} \mathbf{sgn}(\sigma) \phi_1(v \circ (\sigma|I_1)) \phi_2(v \circ (\sigma|I_2)). \tag{8}$$

Proof. Induct on p_2. If $p_2 = 1$ then $\Omega_2 = (-1)^{p_1}\phi_1(w)\phi_2(v_1)$ so (8) holds.

Suppose $p_2 > 1$. Let $J_1 = [\![1, p_1]\!]$ and let $J_2 = (\!(p_1, p_1 + p_2 - 1]\!]$. Arguing inductively we find that

$$\begin{aligned}\Omega_2 &= (-1)^{p_1} \sum_{\rho \in \mathbf{Sh}(J)} \mathbf{sgn}(\rho)\phi_1(w \circ (\rho|J_1))(\phi_2 \llcorner v_1)((w \circ (\rho|J_2)) \\ &= \sum_{\sigma \in \mathbf{Sh}(I), \sigma(1) = p_1 + 1} \mathbf{sgn}(\sigma)\phi_1(v \circ (\sigma|I_1))\phi_2((v \circ (\sigma|I_2)).\end{aligned}$$

□

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$$I \in \mathcal{I}(m, N)$$

$$J \in \mathcal{I}(2, N) \quad J_1 = \bigcup_{i=1}^{m} I_i \quad J_2 = I_{m+1}$$

$$K \in \mathcal{I}(m, N - |I_{m+1}|) \quad K_i = I_i \quad \text{for } i \in [\![1, m]\!]$$

$$\mathbf{g} : \mathbf{Sh}(J) \times \mathbf{Sh}(K) \to \mathbf{Sh}(I)$$

$$\mathbf{g}(\alpha, \beta) = (\alpha \circ \beta) \cup (\alpha|I_{m+1})$$

Lemma 2.3. $\mathbf{rng}\, \mathbf{g} = \mathbf{Sh}(I)$ and

(9) $$\mathbf{sgn}(\mathbf{g}(\alpha, \beta) = \mathbf{sgn}(\alpha)\mathbf{sgn}(\beta) \quad \text{for } \alpha, \beta \in \mathbf{Sh}(J) \times \mathbf{Sh}(K).$$

Moreover,

$$\mathbf{g}(\alpha, \beta)|I_i = (\alpha|J_1) \circ (\beta|K_1) \quad \text{for each } i \in [\![1, m]\!] \text{ and } \quad \mathbf{g}(\alpha, \beta)|I_{m+1} = \alpha|J_2.$$

Proof. Let $\gamma = \mathbf{g}(\alpha, \beta) \in \mathbf{Sh}(I)$. Since $\gamma|I_i$ is increasing for each $i \in [\![1, m+1]\!]$ and since $\mathbf{rng}\, \gamma \subset [\![1, m+1]\!]$ we find that $\gamma \in \mathbf{Sh}(I)$.

Suppose $i, j \in [\![1, m+1]\!]$ and $i < j$. If $j \le m$ we find that

$$\{(k, l) \in I_i \times I_j : \gamma(k) > \gamma(l)\} = \{(k, l) \in I_i \times I_l : \beta(k) > \beta(l)\}$$

and

$$\{(k, l) \in I_i \times I_j : \alpha(k) > \alpha(l)\} = \emptyset$$

since α is increasing on J_1. Also,

$$\{(k, l) \in I_i \times I_{m+1} : \gamma(k) > \gamma(l)\} = \{(k, l) \in I_i \times I_{m+1} : \alpha(\beta(k)) > \alpha(l)\}$$

so, as β permutes J_1,

$$\bigcup_{i=1}^{m} \{(k, l) \in I_i \times I_{m+1} : \gamma(k) > \gamma(l)\} = \bigcup_{i=1}^{m} \{(k, l) \in I_i \times I_{m+1} : \alpha(k) > \alpha(l)\}.$$

Thus $\mathbf{rev}(\gamma)$ is the disjoint union of $\mathbf{rev}(\alpha)$ and $\mathbf{rev}(\beta)$ so (9) holds. □

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Let $\Phi = \bigwedge_{i=1}^m \phi_i$. We have

$$\begin{aligned}\left(\bigwedge_{i=1}^{m+1} \phi_i\right)(v) &= (\Phi \wedge \phi_{m+1})(v) \\ &= \sum_{\alpha \in \mathbf{Sh}(J)} \mathbf{sgn}(\alpha)\Phi(v \circ (\alpha|J_1))\phi_{m+1}(v \circ (\alpha|J_2) \\ &= \sum_{\alpha \in \mathbf{Sh}(J)} \mathbf{sgn}(\alpha)\left(\sum_{\beta \in \mathbf{Sh}(K)} \mathbf{sgn}(\beta)\Pi_{i=1}^m \phi_i(v \circ (\alpha|J_1) \circ (\beta|K_i))\right)\phi_{m+1}(v \circ (\alpha|J_2) \\ &= \end{aligned}$$

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3. Symmetric algebra.

For each $m \in \mathbb{N}$

$$\Xi(E, m)$$

be the set of $\alpha : E \to \mathbb{N}$ such that

$$||\alpha|| = \sum_{a \in E} \alpha(a) = m.$$

Proposition 3.1. Suppose E is finite. Then

$$|\Xi(E, m)| = \binom{m + |E| - 1}{|E| - 1}.$$

Proof. Suppose $\lambda \in \Lambda(|E| - 1 + m, |E| - 1)$. Let $\mathbf{A}(\lambda) \in \Xi(m + |E| - 1, |E| - 1)$ be such that

$$\mathbf{A}(\lambda)(i) = \begin{cases} \lambda(1) - 1 & \text{if } i = 1, \\ \lambda(i) - \lambda(i-1) - 1 & \text{if } i \in (\!(1, |E| - 1]\!], \end{cases}$$

Suppose $\alpha \in \Xi(|E|, m)$. Let $\mathbf{L}(\alpha) \in \Lambda(|E| - 1 + m, |E| - 1)$ be such that

$$\mathbf{L}(\alpha)(i) = i + \sum_{j=1}^{i} \alpha(j) \quad \text{for } i \in [\![1, |E| - 1]\!].$$

Now observe that $\mathbf{A}$ and $\mathbf{L}$ are inverse to one another. □

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$$\mathbf{A}(\mathbf{L}(\alpha))(1) = \mathbf{L}(\alpha)(1) - 1 = 1 + \left(\sum_{j=1}^{1} \alpha(j)\right) - 1 = \alpha(1);$$

If $i > 1$ then

$$\mathbf{A}(\mathbf{L}(\alpha))(i) = \mathbf{L}(\alpha)(i) - \mathbf{L}(\alpha)(i-1) - 1 = i + \sum_{j=1}^{i} \alpha(j) - \left(i - 1 + \sum_{j=1}^{i-1} \alpha(j)\right) - 1 = \alpha(i);$$

$$\mathbf{L}(\mathbf{A}(\lambda))(i) = i + \sum_{j=1}^{i} \mathbf{A}(\lambda)(j) = i + \lambda(1) - 1 + \sum_{j=2}^{i} \lambda(j) - \lambda(j-1) - 1 = \lambda(i).$$

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If $\alpha \in \Xi(E, m)$, $a \in E$ and $\alpha(a) > 0$ we let $\alpha \downarrow a \in \Xi(E, m-1)$ be such that

$$(\alpha \downarrow a)(b) = \begin{cases} \alpha(a) - 1 & \text{if } b = a, \\ \alpha(b) & \text{if } b \in E \sim \{a\}. \end{cases}$$

For $\alpha \in \Xi(E, m)$ we define

$$\mathbf{e}^\alpha \in \bigodot^m V$$

by induction on m by letting

$$\mathbf{e}^\alpha = 1 \quad \text{if } m = 0$$

and, if $m > 0$, by requiring that

$$\mathbf{e}^\alpha = a^* \odot \mathbf{e}^{\alpha \downarrow a} \quad \text{where } a \text{ is the } \prec\text{-first member of } \{b \in E : \alpha(b) \neq 0\}.$$

If $m > 0$ and $\alpha \in \Xi(E, m)$ we define

$$\mathbf{e}_\alpha \in V^m$$

by requiring that

$$\mathbf{e}_\alpha = \overline{a\, \mathbf{e}_{\alpha \downarrow a}} \quad \text{where } a \text{ is the } \prec\text{-first member of } \{b \in E : \alpha(b) \neq 0\}.$$

Proposition 3.2. Suppose $p, q \in \mathbb{N}$. The following statements hold:

(i) if $\alpha \in \Xi(E, p)$ and $b \in E$ then

$$\mathbf{e}^\alpha \llcorner b = 0 \quad \text{if } \alpha(b) = 0;$$

(ii) if $\alpha \in \Xi(E, p)$ and $\beta \in \Xi(E, q)$ then

$$\mathbf{e}^{\alpha+\beta} = \mathbf{e}^\alpha \odot \mathbf{e}^\beta.$$

(iii) If $\alpha \in \Xi(E, p)$, $a \in E$ and $\alpha(a) \neq 0$ then

$$\mathbf{e}^\alpha \llcorner a = \mathbf{e}^{\alpha \downarrow a}$$

and, if $p > 0$,

$$\phi(\mathbf{e}_\alpha) = \phi(\overline{a\, \mathbf{e}_{\alpha \downarrow a}}) \quad \text{for } \phi \in \bigwedge^p V;$$

(iv) if $p > 0$ and $\alpha, \beta \in \Xi(E, p)$ then

$$\mathbf{e}^\alpha(\mathbf{e}_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

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Proposition 3.3. Suppose $p \in \mathbb{N}^+$, $\phi \in \bigodot^p(V, Z)$ and $\phi(\mathbf{e}_\alpha) = 0$ for all $\alpha \in \Xi(P, p)$. Then $\phi = 0$.

Proof. $(\phi \llcorner a)(\mathbf{e}_\beta) = 0$ for all $a \in E$ and $\beta \in \Xi(E, p-1)$ so, inducting on p, we find that

$$\phi(v) = \sum_{a \in E} a^*(v_1)(\phi \llcorner a)(w) = 0.$$

□

Theorem 3.1. Suppose $p \in \mathbb{N}^+$, $\phi \in \bigodot^p(V, Z)$ and $v \in V^p$. Then

$$\phi(v) = \sum_{\alpha \in \Xi(E,p)} \mathbf{e}^\alpha(v) \phi(\mathbf{e}_\alpha).$$

Proof. Both sides have the same value on $\mathbf{e}_\beta$, $\beta \in \Xi(E,p)$. □

Induct on p. This obviously holds if $p = 1$. Suppose $p > 1$. Let $w \in V^{p-1}$ be such that $v = \overline{v_1\, w}$. Then

$$\begin{aligned}\phi(v) &= (\phi \llcorner v_1)(w)\\ &= \sum_{b\in E} b^*(v_1)(\phi\llcorner b)(w)\\ &= \sum_{b\in E} b^*(v_1) \sum_{\beta\in\Xi(E,p-1)} \mathbf{e}^\beta(w)(\phi\llcorner b)(\mathbf{e}_\beta)\\ &= \sum_{b\in E} b^*(v_1) \sum_{\beta\in\Xi(E,p-1)} \mathbf{e}^\beta(w)\phi(\overline{b\,\mathbf{e}_\beta})\end{aligned}$$

Since $\mathbf{e}^\alpha \llcorner b = 0$ if $\alpha \in \Xi(E,p)$ and $\alpha(b) = 0$,

$$\begin{aligned}\sum_{\alpha\in\Xi(E,p)} \mathbf{e}^\alpha(v)\phi(\mathbf{e}_\alpha) &= \sum_{b\in E} b^*(v_1) \sum_{\alpha\in\Xi(E,p)} (\mathbf{e}^\alpha\llcorner b)(w)\phi(\mathbf{e}_\alpha)\\ &= \sum_{b\in E} b^*(v_1) \sum_{\alpha\in\Xi(E,p),\ \alpha(b)>0} (\mathbf{e}^\alpha\llcorner b)(w)\phi(\mathbf{e}_\alpha)\\ &= \sum_{b\in E} b^*(v_1) \sum_{\alpha\in\Xi(E,p),\ \alpha(b)>0} \mathbf{e}^{\alpha\downarrow b}(w)\phi(\overline{b\,\mathbf{e}_{\alpha\downarrow b}}).\end{aligned}$$

4. The covariant exterior product.

For $p \in \mathbb{N}$ we define

$$\wedge_p \in \begin{cases} \mathbb{Lin}(\mathbb{R}, \bigwedge^0(V^*)) & \text{if } p = 0,\\ \mathbb{Lin}(V, \bigwedge^1(V^*)) & \text{if } p = 1,\\ \mathbb{MultiLin}(V^p, \bigwedge^p(V^*)) & \text{if } p = 0,\end{cases}$$

by induction on p as follows. Let $\vartheta : V \to V^{**}$ be as in ??. If $p = 0$ we let $\wedge_p(r) = r$ for $r \in \mathbb{R}$; if $p = 1$ we let $\wedge_p(v) = \vartheta(v)$ for $v \in V$; and if $p > 1$ we require that

$$\wedge_p(v) = \vartheta(v_1) \wedge \wedge_{p-1}(w) \quad \text{if } v \in V^p,\ w \in V^{p-1} \text{ and } v = \overline{v_1\, w}$$

NEW

$$\wedge_p(v) = \wedge^p(\vartheta \circ v)$$

$$\wedge_p(\mathbf{e}_A)(\mathbf{e}^*_B) = \begin{cases} 1 & \text{if } A = B,\\ 0 & \text{if } A \neq B.\end{cases}$$

NEW

Definition 4.1. For $p \in \mathbb{N}$ we let

$$\bigwedge\nolimits_p V = \mathbf{span}\,\{\wedge_p(v) : v \in V^p\}.$$

Proposition 4.1. Suppose $p, q \in \mathbb{N}$, $u \in V^p$ and $v \in V^q$. Then

$$\wedge_p(u) \wedge \wedge_q(v) = \wedge_{p+q}(\overline{u\,v}) \in \bigwedge\nolimits_{p+q} V.$$

Proof. Induct on p. If $t \in V$, $u \in V^p$ and $v \in V^q$ then

$$\begin{aligned}\wedge_{p+1}(\overline{t\,u}) \wedge \wedge_q(v) &= (\vartheta(t) \wedge \wedge_p(u)) \wedge \wedge_q(v) \\ &= \vartheta(t) \wedge (\wedge_p(u) \wedge \wedge_q(v)) \\ &= \vartheta(t) \wedge (\wedge_{p+q}(\overline{u\,v})) \\ &= \wedge_{p+q+1}(\overline{t\,\overline{u\,v}}) \\ &= \wedge_{p+q+1}(\overline{\overline{t\,u}\,v}).\end{aligned}$$

□

Theorem 4.1. Suppose $p, q, r \in \mathbb{N}$. Then

$$\xi \wedge \eta = (-1)^{pq} \eta \wedge \xi \quad \text{for } \xi \in \textstyle\bigwedge_p V \text{ and } \eta \in \textstyle\bigwedge_q V.$$

$$(\xi \wedge \eta) \wedge \zeta = \xi \wedge (\eta \wedge \zeta) \quad \text{for } \xi \in \textstyle\bigwedge_p V,\ \eta \in \textstyle\bigwedge_q V \text{ and } \zeta \in \textstyle\bigwedge_r V.$$

4.1. **Bases.** Suppose E is a basis for V.

Theorem 4.2. Suppose $p \in \mathbb{N}^+$. We have

$$\wedge_p(v) = \sum_{A \subset E,\ |A|=p} \mathbf{e}^A(v) \wedge_p (\mathbf{e}_A) \quad \text{for } v \in V^p.$$

Proof. Induct on p. Obvious if $p = 1$. Suppose $u \in V$ and $v \in V^p$. Arguing inductively we find that

$$\wedge_p(\overline{u\,v}) = \vartheta(u) \wedge \wedge(v)$$

$$\left(\sum_{a \in E} a^*(u)a\right) \wedge \sum_{A \subset E,\ |A|=p} \mathbf{e}^A(v)) \wedge_p (\mathbf{e}_A)$$

$$\begin{aligned}&= \sum_{a \in E} \sum_{A \subset E,\ |A|=p} a^*(u)\mathbf{e}^A(v)\, \vartheta(a) \wedge_p (\mathbf{e}_A) \\ &= \sum_{C \subset E,\ |C|=p+1} \mathbf{e}^C(\overline{u\,v}) \wedge_{p+1} (\mathbf{e}_C)\end{aligned}$$

since, by ??,

$$a^*(u)\mathbf{e}^A(v)\, \vartheta(a) \wedge_p (\mathbf{e}_A) = \begin{cases} be^C(\overline{u\,v}) \wedge_{p+1} (\mathbf{e}_C) & \text{if } a \notin A, \\ 0 & \text{if } a \in A. \end{cases}$$

□

Definition 4.2. If A is a finite subset of E we define

$$\mathbf{e}^*_A \in (V^*)^{|A|}$$

by letting $\mathbf{e}^*_{\{a\}} = a^*$ if $A = \{a\}$ for some $a \in E$ and requiring that

$$\mathbf{e}^*_A = \overline{a^*\, \mathbf{e}^*_{A \sim \{a\}}} \quad \text{if } |A| > 1 \text{ and } a \text{ is the } \prec\text{-first member of } A.$$

Theorem 4.3. Suppose $p \in \mathbb{N}^+$, $A \subset E$, $B \subset E$ and $|A| = p = |B|$. Then

$$\wedge_p(\mathbf{e}_A)(\mathbf{e}^*_B) = \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{if } A \neq B. \end{cases}$$

Proof. Straightforward induction on p. □

Theorem 4.4. $\{\mathbf{e}_p(A) : A \subset E \text{ and } |A| = p\}$ is a basis for $\bigwedge_p V$.

4.1.1. *The universal property of* $\bigwedge_*$.

Definition 4.3. We define

$$\mathbf{M}_{\bigwedge_p V,Z} : \mathbb{Lin}\left(\mathbb{Lin}\left(\bigwedge_p V, Z\right), \bigwedge^p (V,Z)\right)$$

by letting

$$\mathbf{M}_{\bigwedge_p V,Z}(L) = L \circ \wedge_p \quad \text{for } L \in \mathbb{Lin}\left(\bigwedge_p V, Z\right).$$

We let

$$\mathbf{L}_{\bigwedge_p V,Z} = \mathbf{M}^{-1}_{\bigwedge_p V,Z}.$$

Theorem 4.5. We have

$$\mathbf{L}_{\bigwedge_p V,Z} \in \mathbb{Iso}\left(\mathbb{Lin}\left(\bigwedge_p V, Z\right), \bigwedge^p (V,Z)\right).$$

In particular, for any $\mu \in \bigwedge^p(V,Z)$ there is one and only one $L \in \mathbb{Lin}\left(\bigwedge_p V, Z\right)$ such that

$$\mu = L \circ \wedge_p.$$

Moreover, if Y is a vector space and $l \in \mathbb{Lin}(Y,Z)$ then

$$l \circ \mathbf{L}_{\bigwedge_p V,Y}(\mu) = \mathbf{L}_{\bigwedge_p V,Z}(l \circ \mu)$$

for any $\mu \in \bigwedge^p(V,Y)$.

Remark 4.1. In particular,

$$\mathbf{L}_{\bigwedge_p V,\mathbb{R}} \in \mathbb{Iso}\left(\left(\bigwedge_p V\right)^*, \bigwedge^p V\right).$$

Proof. The final assertion of the Theorem is an obvious consequence of the first assertion of the Theorem.

Suppose $L \in \mathbf{ker}\,\mathbf{M}_{\bigwedge_p V,Z}$. Then L vanishes on the range of $\wedge_p$ so L vanishes on $\mathbf{span}\,\mathbf{rng}\,\wedge_p$ and thus equals 0. So $\mathbf{ker}\,\mathbf{M}_{\bigwedge_p V,Z} = \{0\}$.

Let E be a basis for V. Let $\prec$ be a well ordering of E and for $A \subset E$ with $|A| = p$ let $\mathbf{e}^A$ and $\mathbf{e}_A$ be as in ??. Suppose $\mu \in \bigwedge^p(V,Z)$. By ?? and ?? there is $L \in \mathbb{Lin}\left(\bigwedge_p V, Z\right)$ such that $L(\wedge_p(\mathbf{e}_A)) = \mu(\mathbf{e}_A)$ whenever $A \subset E$ and $|A| = p$. It follows that $\mu = \mathbf{M}_{\bigwedge_p V,Z}(L)$ so $\mathbf{rng}\,\mathbf{M}_{\bigwedge_p V,Z} = \bigwedge^p(V,Z)$.

Thus $\mathbf{M}_{\bigwedge_p V,Z} \in \mathbb{Iso}\left(\mathbb{Lin}\left(\bigwedge_p, Z\right), \bigwedge^p(V,Z)\right)$. □

4.2. W**.** Suppose W is a vector space.

Definition 4.4. Suppose $L \in \mathbb{Lin}(V,W)$. We define

$$\bigwedge_p L \in \mathbb{Lin}\left(\bigwedge_p V, \bigwedge_p W\right)$$

by requiring that

$$\left(\bigwedge_p L\right)(\wedge_p(v)) = \wedge_p(w)$$

for $v \in V^p$ and where $w \in W^p$ is such that, for $i \in [\![1,p]\!]$, $w_i = L(v_i)$.

Proposition 4.2. Suppose $l \in \mathbb{L}\text{in}(V, W)$. The following diagram is commutative:

$$\begin{array}{ccc} \bigwedge^p W & \xrightarrow{\bigwedge^p l} & \bigwedge^p V \\ \downarrow \mathbf{L}_{p,W,\mathbb{R}} & & \downarrow \mathbf{L}_{p,V,\mathbb{R}} \\ \left(\bigwedge_p W\right)^* & \xrightarrow{\left(\bigwedge_p l\right)^*} & \left(\bigwedge_p V\right)^* \end{array}$$

Proof. Suppose $\phi \in \bigwedge^p W$, $v \in V^p$ and $w \in W^p$ is such that the i-th coordinate of w, $i \in [\![1, p]\!]$, equals $l(v_i)$. Then

$$\mathbf{L}_{p,V,\mathbb{R}}\left(\left(\bigwedge^p l\right)(\phi)(\wedge_p(v))\right) = \left(\bigwedge^p l\right)(\phi)(v) = \phi(w)$$

and

$$\begin{aligned} \left(\left(\bigwedge_p l\right)^* (\mathbf{L}_{p,W,\mathbb{R}}(\phi))\right)(\wedge_p(v)) &= (\mathbf{L}_{p,W,\mathbb{R}}(\phi))\left(\left(\bigwedge_p l\right)(\wedge_p(v))\right) \\ &= \mathbf{L}_{p,W,\mathbb{R}}(\phi)(\wedge_p(w)) \\ &= \phi(w). \end{aligned}$$

□

Proposition 4.3. Suppose $l \in \mathbb{L}\text{in}(V, W)$. The following diagram is commutative.

$$\begin{array}{ccc} \bigwedge_p (W^*) & \xrightarrow{\bigwedge_p (l^*)} & \bigwedge_p (V^*) \\ \downarrow \wedge^p & & \downarrow \wedge^p \\ \bigwedge^p W & \xrightarrow{\bigwedge^p l} & \bigwedge^p V \end{array}$$

Proof. Suppose $\omega \in (W^*)^p$ and $v \in V^p$. Let $\eta \in (V^*)^p$ is such that its i-th coordinate, $i \in [\![1, p]\!]$, equals $l^*(\omega_i) = \omega_i \circ l \in V^*$. Then

$$\left(\wedge^p \left(\left(\bigwedge_p (l^*)\right)(\wedge_p(\omega))\right)\right)(v) = (\wedge_p(\eta))(v) = \wedge^p(\eta)(v)$$

$$\left(\left(\bigwedge^p l\right)(\wedge_p(\wedge_p(\omega))\right)(v) = \left(\left(\bigwedge^p l\right)(\wedge^p(\omega))\right)(v) = \wedge^p(\eta)(v).$$

□

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Definition 4.5. We let

$$\mathbf{I}_p = \mathbf{L}_{p,V,\mathbb{R}} \circ \wedge^p \in \mathbb{L}\text{in}\left(\bigwedge_p (V^*), \left(\bigwedge_p V\right)^*\right).$$

Proposition 4.4. Suppose $l \in \mathbb{L}\text{in}(V, W)$. The following diagram is commutative.

$$\begin{array}{ccc} \bigwedge_p (W^*) & \xrightarrow{\bigwedge_p (l^*)} & \bigwedge_p (V^*) \\ \downarrow \mathbf{I}_{p,W} & & \downarrow \mathbf{I}_{p,V} \\ \left(\bigwedge_p W\right)^* & \xrightarrow{\left(\bigwedge_p l\right)^*} & \left(\bigwedge_p V\right)^* \end{array}$$

Proof.

$$\begin{aligned}\mathbf{I}_p \circ \left(\bigwedge\nolimits_p (l^*)\right) &= \mathbf{L}_{p,V,\mathbb{R}} \circ \wedge^p \circ \left(\bigwedge\nolimits_p (l^*)\right)\\ &= \mathbf{L}_{p,V,\mathbb{R}} \circ \left(\bigwedge\nolimits^p l\right) \circ \wedge^p\\ &= \left(\bigwedge\nolimits_p l\right)^* \circ \mathbf{L}_{p,W,\mathbb{R}} \circ \wedge^p\\ &= \left(\bigwedge\nolimits_p l\right)^* \circ \mathbf{I}_p.\end{aligned}$$

□

5. Inner Products.

Suppose $\beta \in \mathbb{L}\mathrm{in}(V, V^*)$ is the polarity of an inner product $\bullet$ on V.
For each $p \in \mathbb{N}^+$ and $v \in V^p$ let

$$v^\beta \in (V^*)^p$$

be such that its i-th coordinate, $i \in [\![1, p]\!]$, equals $\beta(v_i)$.

Definition 5.1. For each $p \in \mathbb{N}^+$ let

$$\beta_p = \mathbf{I}_p \circ \left(\bigwedge\nolimits_p \beta\right) \in \mathbb{I}\mathrm{so}\left(\bigwedge\nolimits_p V, \left(\bigwedge\nolimits_p V\right)^*\right).$$

Theorem 5.1. β_p is the polarity of an inner product on $\bigwedge_p V$. In fact,

$$\beta_p(\wedge_p(v))(\wedge_p(w)) = \wedge^p(v^\beta)(w) \quad \text{for } v, w \in V^p.$$

Moreover, if $e \in V^p$ is such that the range of e is an orthonormal basis for V then

$$\{\wedge_p(\mathbf{e}_A) : A \subset [\![1, \mathbf{dim}\, V]\!] \text{ and } |A| = p\}$$

is an orthonormal basis for $\bigwedge_p V$.

Theorem 5.2. Suppose p is an integer not less than 2, $u \in V$, $u \neq 0$, $v \in V^{p-1}$ and $\wedge_{p-1}(v) \neq 0$. Then

$$|\wedge_p (\overline{u\, v})| \leq |u| |\wedge_{p-1} (v)|$$

with equality if and only if $u \in (\mathbf{span}\,\mathbf{rng}\, v)^\perp$.

Proof. Let $s \in \mathbf{rng}\, v$ and $t \in (\mathbf{span}\,\mathbf{rng}\, v)^\perp$ be such that $u = s + t$. Then

$$\begin{aligned}|\wedge_p (\overline{u\, v})|^2 &= (\beta(u) \wedge^{p-1} (\beta(v)))(\overline{u\, v})\\ &= (\beta(u) \wedge^{p-1} (\beta(v)))(\overline{(s+t)\, v})\\ &= ((\beta(u) \wedge^{p-1} (\beta(v))) \llcorner t)(v)\\ &= (\beta(u) \llcorner t) \wedge^{p-1} (\beta(v))(v)\\ &= |u|^2 |\wedge_{p-1} (v)|^2.\end{aligned}$$

□

5.1. **Adjoints.** Suppose W is a finite dimensional inner product space and $l \in \mathbb{Lin}(V, W)$. Then

$$(\bigwedge_p L)^\flat = \beta_{p,W}^{-1} \circ \left(\bigwedge_p L\right)^* \circ \beta_{p,V}.$$

Theorem 5.3.

$$(\bigwedge_p L)^\flat = \bigwedge_p (L^\flat).$$

Proof. Chase through the commutative diagrams. □

5.2. **The Hodge $*$ operator.** Suppose $\mathbf{dim}\, V = n$. Let $\Omega \in \bigwedge_n V$ be such that $|\Omega| = 1$. (Note that the only other member of $\bigwedge_V$ of norm 1 is $-\Omega$.) Let $\Omega^* \in \bigwedge^n V$ be such that $\Omega^*(\Omega) = 1$.

$$\gamma^p : \bigwedge^p V \to \bigwedge_p V$$

be defined by

$$\gamma^p = (\wedge_{V*}^p \circ \bigwedge_p \beta)^{-1}.$$

aldownthrought the inner product. We define

$$\cdot * \in \mathbb{Lin}(\bigwedge_p V, \bigwedge_{n-p} V)$$

by letting

$$*\eta = \gamma^{n-p}(\Omega^* \llcorner \eta).$$

Proposition 5.1. $\cdot^*$ is an isometry. Moreover,

$$\xi \wedge (*\eta) = (\xi \bullet \eta)\Omega$$

and

$$* * \xi = (-1)^{p(n-p)}\xi.$$

Proof. That $\cdot *$ is an isometry can be verified by observing that

$$(*\mathbf{e}_A) \bullet \mathbf{e}_B = \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{if } A \neq B \end{cases}$$

whenever A, B are subsets of E and $|A| = p = |B|$.

We have

$$\Omega^*(\xi \wedge (*\eta) = \Omega^*(\xi \wedge \beta_{n-p}^{-1}(\Omega^* \llcorner \beta_p)(\eta)) = (\Omega^* \llcorner \xi)(\wedge \beta_{n-p}^{-1}(\Omega^* \llcorner \beta_p)) = \xi \bullet \eta.$$

□