1. **Algebras of sets and the integration of elementary functions.**

Let $X$ be a set.

**Definition 1.1.** Suppose $\mathcal{A}$ is a family of subsets of $X$. We let
\[
\mathbf{u}(\mathcal{A}) = \{\cup \mathcal{F} : \mathcal{F} \subset \mathcal{A} \text{ and } \mathcal{F} \text{ is finite}\};
\]
\[
\mathbf{i}(\mathcal{A}) = \{\cap \mathcal{F} : \mathcal{F} \subset \mathcal{A} \text{ and } \mathcal{F} \text{ is finite}\};
\]
\[
\mathbf{c}(\mathcal{A}) = \{X \sim A : A \in \mathcal{A}\}.
\]

We also set
\[
\mathbf{d}(\mathcal{A}) = \{\cup \mathcal{F} : \mathcal{F} \subset \mathcal{A}, \mathcal{F} \text{ is a finite and disjointed}\}.
\]

We say $\mathcal{A}$ is an **algebra of subsets of $X$** if the following three conditions hold:
\[
(U) \quad \mathbf{u}(\mathcal{A}) \subset \mathcal{A}; \quad (I) \quad \mathbf{i}(\mathcal{A}) \subset \mathcal{A}; \quad (C) \quad \mathbf{c}(\mathcal{A}) \subset \mathcal{A}.
\]

Note the redundancy: $U$ and $C$ imply $I$ and $I$ and $C$ imply $U$ by the DeMorgan Laws. Note also that the union of the empty family of subsets of $X$ is the empty set so an algebra of subsets of $X$ always contains $\emptyset$ and $X$.

**Definition 1.2.** Suppose $\mathcal{S}$ is a family of subsets of $X$. Note that $2^X$ is an algebra of subsets of $X$ and that $\mathcal{S} \subset 2^X$ so that
\[
\{A : A \text{ is an algebra of subsets of } X \text{ and } \mathcal{S} \subset A\} \neq \emptyset.
\]

Thus
\[
\mathbf{a}(\mathcal{S}) = \cap\{A : A \text{ is an algebra of subsets of } X \text{ and } \mathcal{S} \subset A\}
\]
is well defined.

**Theorem 1.1.** Suppose $X$ is a set and $\mathcal{S}$ is a family of subsets of $X$. Then
\[
(i) \quad \mathbf{a}(\mathcal{S}) \text{ is an algebra of subsets of } X;
\]
\[
(ii) \quad \mathcal{S} \subset \mathbf{a}(\mathcal{S});
\]
\[
(iii) \quad \text{if } \mathcal{A} \text{ is an algebra of subsets of } X \text{ and } \mathcal{S} \subset \mathcal{A} \text{ then } \mathbf{a}(\mathcal{S}) \subset \mathcal{A}.
\]

**Proof.** This is straightforward once you get used to the definitions.

Suppose $A, B \in \mathbf{a}(\mathcal{S})$. If $\mathcal{A}$ is an algebra of subsets of $X$ and $\mathcal{S} \subset \mathcal{A}$ then $\{A, B\} \subset \mathcal{A}$ so $\{A \cup B, A \cap B, X \sim A\} \subset \mathcal{A}$. Thus $\{A \cup B, A \cap B, X \sim A\} \subset \mathbf{a}(\mathcal{S})$. Thus (i) holds.

Suppose $A \in \mathcal{S}$. If $\mathcal{A}$ is an algebra of subsets of $X$ and $\mathcal{S} \subset \mathcal{A}$ then $A \in \mathcal{A}$. Thus $A \in \mathbf{a}(\mathcal{S})$ so (ii) holds.

(iii) is a direct consequence of the definition of $\mathbf{a}(\mathcal{S})$. \hfill $\square$

So algebras of subsets containing a given family of subsets exist. One calls $\mathbf{a}(\mathcal{S})$ the **algebra of subsets of $X$ generated by $\mathcal{S}$**; in view of (iii) it is the **smallest** algebra of subsets of $X$ containing $\mathcal{S}$.

**Definition 1.3.** Suppose $\mathcal{S}$ is a nonempty family of subsets of $X$ and $\mathcal{R} \subset \mathcal{S}$. Noting that $\mathcal{R} \cup \mathbf{c}(\mathcal{S} \sim \mathcal{R})$ is nonempty we let
\[
\mathbf{b}(\mathcal{R}, \mathcal{S}) = \cap(\mathcal{R} \cap \mathbf{c}(\mathcal{S} \sim \mathcal{R})).
\]

Note that $\mathbf{b}(\mathcal{R}, \mathcal{S}) \in \mathbf{a}(\mathcal{S})$ if $\mathcal{S}$ is finite.

We let
\[
\mathbf{B}(\mathcal{S}) = \{\mathbf{b}(\mathcal{R}, \mathcal{S}) : \mathcal{R} \subset \mathcal{S}\}.
\]
**Proposition 1.1.** Suppose $\mathcal{S}$ is a nonempty family of subsets of $X$ and $\mathcal{R}$ and $\mathcal{C}$ are subfamilies of $\mathcal{S}$. Then

(i) $b(\emptyset, \mathcal{S}) = X \sim \bigcup \mathcal{S}$;

(ii) $b(\mathcal{S}, \mathcal{S}) = \bigcap \mathcal{S}$;

(iii) $b(\mathcal{Q} \cup \mathcal{R}, \mathcal{S}) = b(\mathcal{Q}, \mathcal{S}) \cup b(\mathcal{R}, \mathcal{S})$;

(iv) $b(\mathcal{Q} \cap \mathcal{R}, \mathcal{S}) = b(\mathcal{Q}, \mathcal{S}) \cap b(\mathcal{R}, \mathcal{S})$;

(v) $b(\mathcal{S} \sim \mathcal{R}, \mathcal{S}) = X \sim b(\mathcal{R}, \mathcal{S})$.

Moreover, $B(\mathcal{S})$ is a partition of $X$.

**Remark 1.1.** Thus $2^S \ni \mathcal{R} \mapsto b(\mathcal{R}, \mathcal{S}) \in 2^X$ is a morphism of Boolean algebras.

**Proof.** Exercise for the reader. □

**Theorem 1.2.** Suppose $\mathcal{S}$ is a finite nonempty family of subsets of $X$. Then

$$a(\mathcal{S}) = \{a(F) : F \text{ is a finite subfamily of } \mathcal{S}\}.$$

**Proof.** We leave the proof as a straightforward exercise for the reader. □

**Remark 1.2.** Thus $|a(\mathcal{S})| \leq 2^{2^{|\mathcal{S}|}}$.

Moreover, $S = \bigcup \{R \in B(\mathcal{S}) : R \subset S\}$ whenever $S \in \mathcal{S}$.

**Exercise 1.1.** Draw the typical Venn diagram with three sets $A, B, C$ and picture the members of $a(\{A, B, C\})$.

**Example 1.1.** Suppose $n$ is a positive integer. Let $X = \{0, 1\}^n$ and let

$\mathcal{S} = \{S_i : i \in \{1, \ldots, n\}\}$

where for each $i \in \{1, \ldots, n\}$ we have set

$S_i = \{a \in X : x_i = 1\}$.

Then $|\{b(\mathcal{R}, \mathcal{S}) : \mathcal{R} \subset \mathcal{S}\}| = 2^n$

and

$|a(\mathcal{S})| = 2^{2^n}$

which implies that

$a(\mathcal{S}) = 2^X$.

In this case, the aforementioned morphism of Boolean algebras is an isomorphism.

The preceding Theorem together with The following Proposition provides what could reasonable called a construction of the algebra of sets generated by a family of subsets of $X$.

**Proposition 1.2.** Suppose $\mathcal{S}$ is a family of subsets of $X$. Then

$$a(\mathcal{S}) = \bigcup \{a(\mathcal{F}) : \mathcal{F} \text{ is a finite subfamily of } \mathcal{S}\}.$$ 

**Proof.** Exercise for the reader. □
The next Proposition will be very useful to us.

**Proposition 1.3.** Suppose \( \mathcal{S} \) is a family of subsets of \( X \). Then

(i) \( u(u(\mathcal{S})) = u(\mathcal{S}) \);
(ii) if \( i(\mathcal{S}) \subset \mathcal{S} \) then \( i(u(\mathcal{S})) \subset u(\mathcal{S}) \);
(iii) if \( i(\mathcal{S}) \subset \mathcal{S} \) then \( i(d(\mathcal{S})) \subset d(\mathcal{S}) \);
(iv) If \( i(\mathcal{S}) \subset \mathcal{S} \) and \( c(\mathcal{S}) \subset u(\mathcal{S}) \) then \( a(\mathcal{S}) = a(\mathcal{S}) \);
(v) If \( i(\mathcal{S}) \subset \mathcal{S} \) and \( c(\mathcal{S}) \subset d(\mathcal{S}) \) then \( a(\mathcal{S}) = a(\mathcal{S}) \);

**Proof.** Suppose \( \mathcal{F} \) and \( \mathcal{G} \) are finite subfamilies of \( \mathcal{S} \). Evidently,
\[
(\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G}) = \bigcup (\mathcal{F} \cup \mathcal{G})
\]
so (i) holds. Moreover,
\[
(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \bigcup \{ A \cap B : (A, B) \in \mathcal{F} \times \mathcal{G} \}
\]
so (ii) and (iii) hold.

Suppose \( c(\mathcal{S}) \subset \mathcal{S} \). Then
\[
X \sim \bigcup \mathcal{F} = \cap \{ X \sim A : A \in \mathcal{F} \} \in \begin{cases} u(\mathcal{S}) & \text{if } c(\mathcal{S}) \subset u(\mathcal{S}), \\ d(\mathcal{S}) & \text{if } c(\mathcal{S}) \subset d(\mathcal{S}) \end{cases}
\]
by virtue of (ii) and (iii) so (iv) and (v) hold. \( \square \)

**Definition 1.4.** Suppose \( \mathcal{A} \) is a family of subsets of \( X \). We say a nonnegative extended real valued function \( \mu \) on \( \mathcal{A} \) is **additive** if \( \mu(\emptyset) = 0 \) and
\[
\mu(A \cup B) = \mu(A) + \mu(B) \quad \text{whenever } A, B \in \mathcal{A} \text{ and } A \cap B = \emptyset.
\]

**Example 1.2.** \( 2^\mathcal{S} \ni A \mapsto \vert A \vert \) is an additive. That is, counting is additive

**Proposition 1.4.** (The Inclusion-Exclusion Principle.) Suppose \( \mathcal{S} \) is an algebra of subsets of \( X \) and \( \mu \) is an additive nonnegative extended real valued function on \( \mathcal{A} \). Suppose \( n \) is a positive integer,
\[
S_1, S_2, \ldots, S_n \in \mathcal{A},
\]
and \( \mu(S_i) < \infty \) for \( i \in \{1, 2, \ldots, n\} \). Then
\[
\mu(\bigcup_{i=1}^n S_i) = \sum_{k=1}^n (-1)^k \sum_{\lambda \in \Lambda(k, n)} \mu(\cap_{i=1}^k S_{\lambda(i)})
\]
where \( \Lambda(k, n) \) is the set of increasing maps from \( \{1, 2, \ldots, k\} \) into \( \{1, 2, \ldots, n\} \).

**Proof.** Exercise for the reader. Induct on \( n \). \( \square \)

**Proposition 1.5.** Suppose \( A \) is a sequence of subsets of \( X \). Let \( B_0 = A_0 \) and, for each \( n \in \mathbb{N}^+ \), let
\[
B_n = A_n \sim \bigcup_{j=0}^{n-1} A_j.
\]
Then \( \{B_n : n \in \mathbb{N}\} \) is disjointed and
\[
\bigcup_{n=0}^\infty A_n = \bigcup_{n=0}^\infty B_n.
\]

**Proof.** Exercise for the reader. \( \square \)
Proposition 1.6. Suppose $S$ is an algebra of subsets of $X$, $\mu$ is an additive non-negative extended real valued function on $A$ and $F$ is a finite subfamily of $A$. Then

$$\mu(\bigcup F) \leq \sum_{A \in F} \mu(A).$$

Proof. Exercise for the reader. Make use of the obvious modification of the Proposition 1.5 to the present situation. \hfill \square

Theorem 1.3. Suppose $S$ is a family of subsets of $X$ such that

$$i(S) \subset S \quad \text{and} \quad c(S) \subset d(S)$$

and suppose

$$\gamma : S \rightarrow [0, \infty]$$

is such that

$$\gamma(S) = \sum_{T \in F} \gamma(T)$$

whenever $S \in S$ and $F$ is a finite disjointed subfamily of $S$ with union $S$.

Then there is one and only one nonnegative extended real valued additive function $\mu$ on $a(S)$ such that

$$\mu(S) = \gamma(S) \quad \text{whenever} \quad S \in S.$$

Remark 1.3. Note that if $\emptyset \in S$ then $\gamma(\emptyset) = 0$.

Proof. By the preceding Theorem we have

(i) \hfill $a(S) = d(S)$.

Let

$$\mu = \{(\bigcup F, \sum_{T \in F} \gamma(T)) : F \text{ is a finite disjointed subfamily of } S\}.$$

Then the domain of $\mu$ equals $a(S)$ by (i).

Suppose $F$ and $G$ are finite disjointed subfamilies of $S$ and $\bigcup F = \bigcup G$. Then

$$\sum_{T \in F} \gamma(T) = \sum_{T \in F} \sum_{G \in G} \gamma(F \cap G)$$

and

$$\sum_{U \in G} \gamma(U) = \sum_{U \in G} \sum_{T \in F} \gamma(G \cap F)$$

so $\mu$ is a function. Moreover, $(\emptyset, 0) \in \mu$.

Suppose $A, B \in a(S) \sim \{\emptyset\}$ and $A \cap B = \emptyset$. By (i) there are finite disjointed subfamilies $F$ and $G$ of $S$ such that $\emptyset \notin F \cup G$, $A = \bigcup F$ and $B = \bigcup G$. Then $F \cup G$ is a finite disjointed subfamily of $S$ with union $A \cup B$ and since $F \cap G = \emptyset$ we find that

$$\mu(A \cup B) = \sum_{V \in F \cup G} \gamma(V) = \sum_{T \in F} \gamma(T) + \sum_{U \in G} \gamma(U) = \mu(A) + \mu(B).$$

\hfill \square
1.1. Multiintervals.

**Definition 1.5.** Let \( \mathcal{I} \) be the family of intervals in \( \mathbb{R} \). Let \( \mathcal{M} \) be the family of finite unions of intervals in \( \mathbb{R} \); we call the members of \( \mathcal{M} \) multiintervals.

In the terminology introduced earlier, 
\[
\mathcal{M} = \text{u}(\mathcal{I}).
\]

**Definition 1.6.** Whenever \( I \in \mathcal{I} \) we let 
\[
I^- = \{ y \in \mathbb{R} : y < x \text{ whenever } x \in I \} \quad \text{and} \quad I^+ = \{ y \in \mathbb{R} : x < y \text{ whenever } x \in I \}.
\]

Note that if \( I = \emptyset \) then \( I^- = \mathbb{R} = I^+ \) and that if \( I = \mathbb{R} \) then \( I^- = \emptyset \) and \( I^+ = \emptyset \). Be careful!

**Proposition 1.7.** Suppose \( I \in \mathcal{I} \) and \( I \neq \emptyset \). Then
\[
\text{(i) } I^-, I^+ \in \mathcal{I}; \\
\text{(ii) } \{I^-, I, I^+\} \text{ is a partition of } \mathbb{R}; \\
\text{(iii) } \sup I^- = \inf I \text{ and } \sup I = \inf I^+; \\
\text{(iv) } \text{if } J \in \mathcal{I} \text{ and } I \cap J = \emptyset \text{ then either } J \subset I^- \text{ or } J \subset I^+.
\]

**Proof.** (i) is obvious.

Suppose \( z \in \mathbb{R} \sim (I^- \cup I^+) \). Then there are \( x \in I \) and \( y \in I \) such that \( x < z < y \). Since \( I \) is an interval we infer that \( x \in I \). Thus \( \mathbb{R} = \cup \{I^-, I, I^+\} \). That \( \{I^-, I, I^+\} \) is disjointed is evident. Thus (ii) holds.

Suppose \( y \in I \). Then \( y \) is an upper bound for \( I^- \) so \( \sup I^- \leq y \). Thus \( \sup I^- \) is a lower bound for \( I \) so \( \sup I^- \leq \inf I \). Suppose, contrary to (iii), \( \sup I^- < \inf I \). Then there would be be \( x \in \mathbb{R} \) such that \( \sup I^- < x < \inf I \). But \( x < \inf I \) implies \( x \in I^- \) which would imply \( x \leq \sup I^- \). Thus \( \sup I^- = \inf I \). In a similar fashion one shows that \( \sup I = \inf I^+ \).

Suppose \( J \in \mathcal{I} \), \( x \in J \cap I^- \) and \( z \in J \cap I^+ \). Suppose \( y \in I \). Then, as \( J \) is an interval and \( x < y < z \) we have \( y \in J \) which implies \( I \subset J \). Thus (iv) holds. \( \square \)

**Corollary 1.1.** Suppose \( \mathcal{J} \) is a disjointed family of nonempty intervals. Then
\[
\{ (J, K) \in \mathcal{J} \times \mathcal{J} : J \subset K^- \}
\]
is a linear ordering of \( \mathcal{J} \).

**Proposition 1.8.** \( \mathcal{M} \) is an algebra of subsets of \( \mathbb{R} \).

**Proof.** Note that the intersection of a nonempty family of intervals is an interval; that is, \( \text{i}(\mathcal{I}) \subset \mathcal{I} \). Suppose \( I, J \in \mathcal{I} \). Then \( I \cap J \in \mathcal{I} \) and \( I \sim J = (I \cap J^-) \cup (I \cap J^+) \in \mathcal{M} \); thus \( \text{c}(\mathcal{I}) \subset \text{d}(\mathcal{I}) \). Now apply (iv) and (iv) of Proposition 1.3 \( \square \)

**Proposition 1.9.** A subset of \( \mathbb{R} \) is a multiinterval if and only if it has finitely many connected components.

**Proof.** Suppose \( A \subset \mathbb{R} \) and let \( \mathcal{C} \) be the family of connected components of \( A \). Since \( \mathcal{C} \subset \mathcal{I} \) we have that \( A \in \mathcal{M} \) if \( \mathcal{C} \) is finite.

Suppose \( A = \cup \mathcal{F} \) where \( \mathcal{F} \) is a finite subfamily of \( \mathcal{I} \). Let 
\[
c = \{ (I, C) : I \in \mathcal{F}, C \in \mathcal{C} \text{ and } I \subset C \}.
\]
Suppose \( I \in \mathcal{F} \). There is one and only one \( C \in \mathcal{C} \) such that \( I \cap C \neq \emptyset \). Since \( I \cap C \) is connected we find that \( I \cup C \) is connected; thus \( I \subset C \). That is, \( c \) is a function, \( \text{dmc} c = \mathcal{F} \) and \( \text{rng} c = \mathcal{C} \). Thus \( \mathcal{C} \) is finite since it is the range of a function with a finite domain. \( \square \)
**Proposition 1.10.** Suppose $I$ and $J$ are intervals and $I \cap J \neq \emptyset$. Then $I \cup J$ is an interval.

*Proof.* \{I, J\} is a family of connected sets with a nonempty intersection so its union is connected. □

**Proposition 1.11.** Suppose $\mathcal{J}$ finite family of intervals and $\mathcal{K}$ is the family of connected components of members of $B(\mathcal{J})$ which are subsets of $\cup \mathcal{J}$. Then $\mathcal{K}$ is a finite disjointed family of intervals, $\cup \mathcal{J} = \cup \mathcal{K}$ and

\[ J = \cup \{K \in \mathcal{K} : K \subset J\} \quad \text{whenever } J \in \mathcal{J}. \]

*Proof.* We have shown that

\[ J = \cup \{S \in B(\mathcal{J}) : S \subset J\}. \]

So the Proposition follows from the fact that $B(\mathcal{J})$ is disjointed and the fact that the number of connected components of a member of $B(\mathcal{J})$ is finite since $B(\mathcal{J}) \subset \mathcal{M}$. □

We now proceed to construct the unique additive nonnegative extended real valued function

\[ || \cdot || \]

on $\mathcal{M}$ such that $||I|| = \text{diam} I$ whenever $I$ is a nonempty interval.

**Lemma 1.1.** Suppose $I \in \mathcal{I} \sim \{\emptyset\}$, $\mathcal{J}$ is a finite disjointed family of nonempty intervals and $I = \cup \mathcal{J}$. Then

\[ \text{diam} I = \sum_{J \in \mathcal{J}} \text{diam} J. \]

*Proof.* We may assume that $\text{diam} I < \infty$. We prove this by induction on the number of members of $\mathcal{J}$. The assertion holds trivially if $\mathcal{J}$ has one member so suppose $\mathcal{J}$ has at least two members and that the Lemma holds for families having fewer members than $\mathcal{J}$.

Let

\[ \lambda = \{(J, K) : J, K \in \mathcal{J} \text{ and } J \subset K^{-}\} \]

and note that, by Corollary 1.1, $\lambda$ linearly orders on $\mathcal{J}$. Let $K$ be the unique $\lambda$-largest element of $\mathcal{J}$ and let $\mathcal{K} = \mathcal{J} \sim \{K\}$. Then

\[ I \sim K = \cup \mathcal{J} \subset K^{-} \]

which implies that $\sup(I \sim K) \leq \inf K$. Since $I$ is an interval and $I \sim K \neq \emptyset$ we infer that

\[ \sup(I \sim K) = \inf K. \]

It is also clear that

\[ \inf I = \inf(I \sim K) \quad \text{and} \quad \sup I = \sup K. \]

Suppose $x, z \in I \sim K$ and $x < y < z$. Since $I$ is an interval we infer that $y \in I$. Since $y < z \in K^{-}$ we infer that $y \in K^{-} \subset \mathbb{R} \sim K$. Thus $y \in I \sim K$ and, therefore, $I \sim K$ is an interval.
Using the inductive hypothesis on \( \mathcal{J} \) we find that
\[
\text{diam } I = \sup I - \inf I = (\sup K - \inf K) + (\sup(I \sim K) - \inf(I \sim K))
\]
\[
= \text{diam } K + \sum_{J \in \mathcal{J}} \text{diam } J
\]
\[
= \sum_{J \in \mathcal{I}} \text{diam } J
\]
as desired. \( \square \)

**Theorem 1.4.** There is one and only one additive function
\[
|| \cdot ||
\]
on the algebra of multiintervals such that
\[
||I|| = \text{diam } I \quad \text{whenever } I \text{ is a nonempty interval in } \mathbb{R}.
\]

**Proof.** since \( i(I) \subset I \) and \( c(I) \subset d(I) \) the Theorem follows from Lemma 1.1 and Theorem 1.3. \( \square \)

2. **Integration of elementary functions.**

Suppose \( X \) is a set, \( \mathcal{A} \) is an algebra of subsets of \( X \) and \( \mu \) is an additive nonnegative extended real value function on \( \mathcal{A} \).

**Definition 2.1.** We say a function \( s \) with domain \( X \) is \( \mathcal{A} \)-elementary if the range of \( s \) is finite and \( s^{-1}([y]) \in \mathcal{A} \) whenever \( y \in \text{rng } s \). If \( Y \) is a set we let
\[
\mathcal{S}(\mathcal{A}, Y)
\]
be the set of \( \mathcal{A} \)-elementary functions \( s \) such that \( \text{rng } s \subset Y \).

We let
\[
\mathcal{S}^+(\mathcal{A}) = \mathcal{S}(\mathcal{A}, [0, \infty])
\]
and we let
\[
\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{A}, \mathbb{R}).
\]

**Remark 2.1.** Note that if \( Y \) is a set and \( f : X \to Y \) then
\[
\{f^{-1}([y]) : y \in Y\}
\]
is a partition of \( X \).

**Definition 2.2.** For each \( s \in \mathcal{S}^+(\mathcal{A}) \) we set
\[
I^+_\mu(s) = \sum_{y \in \text{rng } s} \mu(s^{-1}([y]))y.
\]

**Proposition 2.1.** The following statements hold:

(i) If \( A \in \mathcal{A} \) then \( 1_A \in \mathcal{S}^+(\mathcal{A}) \) and \( I^+_\mu(1_A) = \mu(A) \).

(ii) If \( s, t \in \mathcal{S}^+(\mathcal{A}) \) and \( s \leq t \) then \( I^+_\mu(s) \leq I^+_\mu(t) \).

(iii) If \( c \in [0, \infty] \) and \( s \in \mathcal{S}^+(\mathcal{A}) \) then \( cs \in \mathcal{S}^+(\mathcal{A}) \) and \( I^+_\mu(cs) = cI^+_\mu(s) \).

(iv) If \( s, t \in \mathcal{S}^+(\mathcal{A}) \) then \( s + t \in \mathcal{S}^+(\mathcal{A}) \) and \( I^+_\mu(s + t) = I^+_\mu(s) + I^+_\mu(t) \).
Proof. We leave the simple proofs of (i) and (iii) to the reader.
Suppose $s, t \in S^+(A)$. Whenever $y \in [0, \infty]$ we set $A_y = s^{-1}\{y\}$ and we set $B_y = t^{-1}\{y\}$. Using the additivity of $\mu$ we find that

\begin{equation}
I^+_{\mu}(s) = \sum_{y \in \text{rng } s} \mu(A_y) y = \sum_{y \in \text{rng } s} \sum_{z \in \text{rng } z} \mu(A_y \cap B_z) z = \sum_{(y,z) \in \text{rng } s \times \text{rng } t} \mu(A_y \cap B_z) y;
\end{equation}

\begin{equation}
I^+_{\mu}(t) = \sum_{z \in \text{rng } t} \mu(B_z) z = \sum_{z \in \text{rng } t} \sum_{y \in \text{rng } y} \mu(A_y \cap B_z) z = \sum_{(y,z) \in \text{rng } s \times \text{rng } t} \mu(A_y \cap B_z) z.
\end{equation}

If $s \leq t$ and $y, z \in [0, \infty]$ are such that $A_y \cap B_z \neq \emptyset$ then $y \leq z$ which implies (ii).

For any non-negative real valued number $w$ we have

$$(s + t)^{-1}\{w\} = \cup \{A_y \cap B_z : y + z = w\}.$$ 

In particular, it follows that $s + t$ is $A$-elementary and

\begin{equation}
I^+_{\mu}(s + t) = \sum_{w \in \text{rng } s + t} \mu(\cup\{A_y \cap B_z : y + z = w\}) w
= \sum_{w \in \text{rng } s + t} \sum_{y + z = w} \mu(A_y \cap B_z) w
= \sum_{(y,z) \in \text{rng } s \times \text{rng } t} \mu(A_y \cap B_z) (y + z).
\end{equation}

That $I^+_{\mu}(s + t) = I^+_{\mu}(s) + I^+_{\mu}(t)$ now follows from (1). \qed

**Proposition 2.2.** Suppose $Y$ is a vector space. Then

$S(A,Y)$ is a linear subspace of $Y^X$.

**Proof.** It is obvious that $S(A,Y)$ is closed with respect to scalar multiplication. Suppose $s, t \in S(A,Y)$ and $w \in \text{rng } s + t$; then

$$(s + t)^{-1}\{w\} = \cup \{A_y \cap B_z : y + z = w\} \in A$$

so $S(A,Y)$ is closed with respect to addition as well. \qed

**Definition 2.3.** Suppose $Y$ is a vector space and $s \in S(A,Y)$. We say $s$ is $\mu$-integrable if $\mu(s^{-1}\{y\}) < \infty$ whenever $y \in Y \sim \{0\}$ in which case we set

\begin{equation}
I_{\mu,Y}(s) = \sum_{y \in \text{rng } s \sim \{0\}} \mu(s^{-1}\{y\}) y.
\end{equation}

We let

$$I_{\mu,Y} = I_{\mu,Y}.$$
Proof. Proceed as in the proof of the preceding Proposition. □

3. Products.

Suppose $X$ and $Y$ are sets; $A$ is an algebra of subsets of $X$; and $B$ is an algebra of subsets of $Y$.

**Definition 3.1.** Whenever $C \subset A \times B$ we set

$$r(C) = \{ A \times B : (A, B) \in C \}.$$  

We let

$$a(A, B) = a(r(A \times B)).$$

**Theorem 3.1.** We have

(i) $i(r(A \times B)) \subset r(A \times B);$  
(ii) $c(r(A \times B)) \subset d(r(A \times B));$  
(iii) $a(A, B) = d(r(A \times B)).$

**Proof.** Suppose $A, C \in A$ and $B, D \in B$. Then

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \in r(A \times B)$$

so (i) holds and

$$(A \times B) \sim (C \times D) = ((A \cap C) \times (B \sim D)) \cup ((A \sim C) \times D) \in d(A \times B)$$

so (ii) holds. Finally, (iii) follows from Proposition 1.3 (v). □

**Theorem 3.2.** There is one and only one additive nonnegative extended real valued function $\mu \otimes \nu$ on $a(A, B)$ such that

$$\mu \otimes \nu(A \times B) = \mu(A)\nu(B) \quad \text{whenever} \quad (A, B) \in A \times B.$$  

**Proof.** Suppose $C \in a(A, B)$.

By the preceding Theorem there is a finite subset $C$ of $A \times B$ such that $r(C)$ is disjointed with union $C$. For each $x \in X$ we have

$$Y \ni y \mapsto 1_C(x, y) = \sum_{(A, B) \in C} 1_A(x)1_B \in S^+(B)$$

so that

$$I^+_\nu(Y \ni y \mapsto 1_C(x, y)) = \sum_{(A, B) \in C} \nu(B)1_A(x).$$

Thus

$$X \ni x \mapsto I^+_\mu(Y \ni y \mapsto 1_C(x, y)) = \sum_{(A, B) \in C} \nu(B)1_A \in S^+(A).$$

So we let

$$\mu \otimes \nu(C) = I^+_\mu(X \ni x \mapsto I^+_\nu(Y \ni y \mapsto 1_C(x, y))).$$

It is an immediate consequence of this definition that

$$\mu \otimes \nu(A \times B) = \mu(A)\nu(B) \quad \text{whenever} \quad (A, B) \in A \times B.$$  

We leave to the reader the straightforward verification that $\mu \otimes \nu$ is additive. □

**Remark 3.1.** Alternatively, we could have let $\nu \otimes \mu(C)$ be

$$I^+_\nu(Y \ni y \mapsto I^+_\mu(X \ni x \mapsto 1_C(x, y))).$$
Theorem 3.3. Suppose $u$ is $a(A, B)$-elementary with values in $[0, \infty]$. Then

(i) for each $y \in Y$, $X \ni x \mapsto u(x, y)$ is $A$-elementary;
(ii) $Y \ni y \mapsto I_\mu(X \ni x \mapsto u(x, y))$ is $B$-elementary;
(iii) for each $x \in X$, $Y \ni y \mapsto u(x, y)$ is $B$-elementary;
(iv) $X \ni x \mapsto I_\nu(Y \ni y \mapsto u(x, y))$ is $A$-elementary;
(v) $I_\nu(Y \ni y \mapsto I_\mu(X \ni x \mapsto u(x, y))) = I_{\mu \oplus \nu}(u) = I_\mu(X \ni x \mapsto I_\nu(Y \ni y \mapsto u(x, y))))$.

Moreover, if $Z$ is a vector space and $v$ is a $\mu \otimes \nu$ integrable function with values in $Z$ then

(vi) $X \ni x \mapsto v(x, y)$ is $\mu$-integrable for each $y \in Y$;
(vii) $Y \ni y \mapsto I_\mu(X \ni x \mapsto v(x, y))$ is $\nu$-integrable;
(viii) $Y \ni y \mapsto v(x, y)$ is $\nu$-integrable for each $x \in X$;
(ix) $X \ni x \mapsto I_\nu(Y \ni y \mapsto v(x, y))$ is $\mu$-integrable;
(x) $I_\nu(Y \ni y \mapsto I_\mu(X \ni x \mapsto v(x, y))) = I_{\mu \oplus \nu}(v) = I_\mu(X \ni x \mapsto I_\nu(Y \ni y \mapsto v(x, y))))$.

Proof. First verify that (i)-(v) hold when $u = 1_{A \times B}$ for some $(A, B) \in \mathcal{A} \times \mathcal{B}$. Next verify that the set of functions for which (i)-(v) hold closed under addition and multiplication by nonnegative scalars. Finally, note that if $u \in \mathcal{S}^+(a(A, B))$ we may choose for each $y \in \text{rng } u$ a subset $U_y$ of $A \times B$ such that $r(U_y)$ is disjointed and $\{u = y\} = \cup r(U_y)$. It follows that

$$u = \sum_{y \in \text{rng } u} \sum_{C \in r(U_y)} y1_C.$$

Thus (i)-(v) hold.

One may use the same technique to prove (vi)-(x). \qed

3.1. Another approach. Let

$$p : X \times Y \to X \quad \text{and} \quad q : X \times Y \to Y$$

carry $(x, y) \in X \times Y$ to $x$ and $y$, respectively.

Theorem 3.4. (Product Decomposition Theorem.) Suppose $C$ is a nonempty family of subsets of $X \times Y$,

$$D = \mathcal{B}(|\{x : (x, y) \in C\} : y \in Y \quad \text{and} \quad C \in C|)$$

and

$$E = \mathcal{B}(|\{y : (x, y) \in C\} : x \in X \quad \text{and} \quad C \in C|).$$

Then

$$C = \cup \{D \times E : (D, E) \in D \times E \quad \text{and} \quad D \times E \subset C\} \quad \text{whenever} \ C \in C.$$

Proof. Suppose $C \in C$. For any $y \in Y$ we have

(1) $\{x : (x, y) \in C\} = \cup \{D \in D : D \subset \{x : (x, y) \in C\}\}$

and for any $x \in X$ we have

(2) $\{y : (x, y) \in C\} = \cup \{E \in E : E \subset \{y : (x, y) \in C\}\}.$

Keeping in mind that $D$ is a partition of $X$, for each $x \in X$ we let $D_x$ be such that $x \in D_x \in \mathcal{D}$. Keeping in mind that $E$ is a partition of $Y$, for each $y \in Y$ we let $E_y$ be such that $y \in E_y \in \mathcal{E}$. Therefore
Suppose \((a, b) \in C\). We will show that
\[ A_a \times B_b \subset C \]
and that will complete the proof. So suppose \((c, d) \in A_a \times B_b\). From (1) we infer that
\[ A_a \subset \{x : (x, b) \in C\}. \]
From (3) we infer that \((c, b) \in C\). From (2) we infer that
\[ B_b \subset \{y : (c, y) \in C\}. \]
But, as \(d \in B_b\), (4) implies that \((c, d) \in C\) as desired. \(\square\)

**Theorem 3.5.** Suppose \(C\) is a nonempty finite subfamily of \(a(\mathcal{A}, \mathcal{B})\),
\[ D = B(\{\{x : (x, y) \in C\} : y \in Y \text{ and } C \in \mathcal{C}\}) \]
and
\[ E = B(\{\{y : (x, y) \in C\} : x \in X \text{ and } C \in \mathcal{C}\}). \]
Then \(D\) and \(E\) are finite subfamilies of \(\mathcal{A}\) and \(\mathcal{B}\), respectively, and
\[ C = \cup\{D \times E : (D, E) \in D \times E \text{ and } D \times E \subset C\} \quad \text{whenever } C \in \mathcal{C}. \]

**Proof.** This follows easily from the Product Decomposition Theorem. \(\square\)

**Lemma 3.1.** Suppose
(i) \(C \subset A \times B\);
(ii) \(r(C)\) is disjointed;
(iii) \(C = A \times B\) where \(C = \cup r(C), A = p[C]\) and \(B = q[C]\).
Then
\[ \mu(A)\nu(B) = \sum_{(F,G) \in \mathcal{C}} \mu(F)\nu(G). \]

**Proof.** We give two proofs. (At least one will be correct, right?)

**Proof One.** We induct on \(|C|\). The Theorem obviously holds when \(N = 1\). So suppose \(N \in \mathbb{N}^+, N \geq 2\); and the Theorem holds for any \(M \in \mathbb{N}^+\) with \(M < N\); and \(N = |C|\).
Choose \((C, D) \in C\). For each subset \(E\) of \(C\) let
\[ c_1(E) = E \cap C \quad \text{and let} \quad c_0(E) = E \sim C; \]
For each subset \(F\) of \(C\) let
\[ d_1(F) = F \cap D \quad \text{and let} \quad d_0(F) = F \sim D. \]
Evidently,
\[ \mu(E)\nu(F) = \mu(c_1(E))\nu(d_1(F)) + \mu(c_1(E))\nu(d_0(F)) + \mu(c_0(E))\nu(d_1(F)) + \mu(c_0(E))\nu(d_0(F)) \]
(2)
whenever \((E, F) \in \mathcal{C}\). Let
\[
\mathcal{C}_{1,1} = \{(C, D)\} = \{c_1(C), d_1(D)\};
\]
\[
\mathcal{C}_{1,0} = \{(c_1(E), d_0(F)) : (E, F) \in \mathcal{C} \sim \{(C, D)\}\};
\]
\[
\mathcal{C}_{0,1} = \{(c_0(E), d_1(F)) : (E, F) \in \mathcal{C} \sim \{(C, D)\}\};
\]
\[
\mathcal{C}_{0,0} = \{(c_0(E), d_0(F)) : (E, F) \in \mathcal{C} \sim \{(C, D)\}\}.
\]
Since for each \((i, j) \in \{0, 1\} \times \{0, 1\}\) we have
\[
|\mathcal{C}_{i,j}| < N \quad \text{and} \quad \mathcal{C}_{i,j}\text{ is disjointed}
\]
and since
\[
C \times D = \bigcup \mathcal{C}_{1,1};
\]
\[
C \times (B \sim D) = \bigcup \mathcal{C}_{1,0};
\]
\[
(A \sim C) \times D = \bigcup \mathcal{C}_{0,1};
\]
\[
(A \sim C) \times (B \sim D) = \bigcup \mathcal{C}_{0,0}
\]
we find that
\[
\mu(C)\nu(D) = \sum_{(E, F) \in \mathcal{C}_{1,1}} c_1(E)d_1(F);
\]
\[
C \times (B \sim D) = \sum_{(E, F) \in \mathcal{C}_{1,0}} c_1(E)d_0(F);
\]
\[
(A \sim C) \times D = \sum_{(E, F) \in \mathcal{C}_{0,1}} c_0(E)d_1(F);
\]
\[
(A \sim C) \times (B \sim D) = \sum_{(E, F) \in \mathcal{C}_{0,0}} c_0(E)d_0(F);
\]
here we need to make use of the fact that if \(P, Q, R, S\) are sets and \(P \times Q = R \times S\) then either \(P = R\) and \(Q = S\) or at least one of \(P, Q\) and one of \(R, S\) are empty. Consequently,
\[
\mu(A)\nu(B) = (\mu(C) + \mu(A \sim C))(\nu(D) + \mu(B \sim D))
\]
\[
= \mu(C)\nu(D) + \mu(C)\nu(B \sim D) + \mu(A \sim C)(\nu(D) + \nu(A \sim C)\mu(B \sim D))
\]
\[
= \sum_{(i, j) \in \{0, 1\} \times \{0, 1\}} \sum_{(E, F) \in \mathcal{C}_{i,j}} \mu(c_i(E))\nu(d_j(F))
\]
\[
= \sum_{(E, F) \in \mathcal{C}} \sum_{(i, j) \in \{0, 1\} \times \{0, 1\}} \mu(c_i(E))\nu(d_j(F))
\]
\[
= \sum_{(E, F) \in \mathcal{C}} \mu(E)\nu(F).
\]

**Proof Two.** Uses Product Decomposition Theorem. Let \(D = B(\{D : (D, E) \in \mathcal{C}\})\) and let \(E = B(\{E : (D, E) \in \mathcal{C}\})\). Let \(D' = \{D \in D : D \subset A\}\) and let \(E' = \{E \in E : E \subset B\}\). Since \(p(C) = A\) we find that \(D'\) is a partition of \(A\) and since \(p(C) = B\) we find that \(E'\) is a partition of \(B\). Thus
\[
\mu(A)\nu(B) = \left(\sum_{D \in D'} \mu(D)\right)\left(\sum_{E \in E'} \mu(E)\right) = \sum_{(D, E) \in D'} \mu(D)\nu(E).
\]
For each \((F, G) \in \mathcal{C}\) we note that \(\{D \in \mathcal{D'} : D \subset F\}\) is a partition of \(F\) and \(\{E \in \mathcal{E'}\}\) is a partition of \(G\). Thus

\[
\sum_{(F, G) \in \mathcal{C}} \mu(F) \nu(G) = \sum_{(F, G) \in \mathcal{C}} \left( \sum_{D \in \mathcal{D'}, \ D \subset F} \mu(F) \right) \left( \sum_{E \in \mathcal{E'}, \ E \subset G} \nu(G) \right)
\]

\[
= \sum_{(F, G) \in \mathcal{C}} \left( \sum_{(D, E) \in \mathcal{D'} \times \mathcal{E'}, \ D \subset F, E \subset G} \mu(F) \nu(G) \right)
\]

Since \(r(\mathcal{C})\) is disjointed we find with the help of a preceding Theorem that \(\mathcal{D'} \times \mathcal{E'}\) is the disjointed union of\(\{(D, E) \in \mathcal{D'} \times \mathcal{E'} : D \subset F\text{ and }E \subset G\} : (F, G) \in \mathcal{C}\).

\[\square\]

**Multirectangles.**

Let \(n\) be a positive integer.

If \(n > 1\), for each \(j \in \{1, \ldots, n\}\) we let

\[p_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}\text{ and }q_j : \mathbb{R}^n \rightarrow \mathbb{R}\]

be such that, for each \(k \in \{1, \ldots, n-1\}\),

\[(p_j(x))_k = \begin{cases} x_k & \text{if } 1 \leq k < j, \\ x_{k+1} & \text{if } j \leq k \leq n-1. \end{cases}\]

and such that

\[q_j(x) = x_j\]

whenever \(x \in \mathbb{R}^n\).

**Definition 3.2.** We say a subset \(R\) of \(\mathbb{R}^n\) is a **rectangle** if either \(n = 1\) and \(R \in \mathcal{M}\) or \(n > 1\) and

\[q_j[R] \in \mathcal{I}\text{ whenever }j \in \{1, \ldots, n\}\]

and

\[R = \prod_{j=1}^{n} q_j[R];\]

equivalently, \(R\) is a rectangle if and only if there are intervals \(I_j, j \in \{1, \ldots, n\}\), such that

\[R = \prod_{j=1}^{n} I_j.\]

Note that this representation is unique if and only if \(R \neq \emptyset\). We say a subset \(M\) of \(\mathbb{R}^n\) is a **multirectangle** if it is the union of a finite family of rectangles. We let

\[\mathcal{R}_n\]

be the family of rectangles in \(\mathbb{R}^n\) and we let

\[\mathcal{M}_n\]

be the family of multirectangles in \(\mathbb{R}^n\). Thus

\[\mathcal{M}_n = \mathcal{u}(\mathcal{R}_n)\].
**Theorem 3.6.** $\mathcal{M}_n$ is an algebra of subsets of $\mathbb{R}^n$ and 

$$\mathcal{M}_n = d(\mathcal{R}_n).$$

Moreover, there is a unique additive nonnegative extended real valued function 

$$|| \cdot ||_n$$

on $\mathcal{M}_n$ such that 

$$||R||_n = \prod_{i=1}^{n} \text{diam } I_i$$

whenever $I_i$, $i = 1, \ldots, n$, are nonempty intervals and $R = \prod_{i=1}^{n} I_i$.

**Proof.** We induct on $n$. Let $T$ be the set of positive integers $n$ for which the Theorem holds. We have already shown that $1 \in T$.

Suppose $n$ is a positive integer and $n \in T$. By our results on products we infer that 

$$a(\mathcal{M}_n, \mathcal{M}) = d(r(\mathcal{M}_n, \mathcal{M})) = d(\mathcal{R}_n, \mathcal{I})$$

and that there is one and only one nonnegative extended real valued additive function 

$$|| \cdot ||_n \otimes || \cdot ||$$

on $a(\mathcal{M}_n, \mathcal{M})$ such that 

$$|| \cdot ||_n \otimes || \cdot ||(M \times N) = ||M||_n||N||$$

whenever $M \in \mathcal{M}_n$ and $N \in \mathcal{M}$. 

$\square$

We let 

$$S^+_n = S^+_{\mathbb{R}^n}$$

and we let 

$$I^+_n = I^+_{|| \cdot ||_n}.$$ 

If $Y$ is a set we let 

$$S_n(Y) = S(\mathcal{M}_n, Y).$$

If $Y$ is a vector space we let 

$$I_{n,Y} = I_{|| \cdot ||_n, Y}.$$ 

**3.2. Product integration on $\mathbb{R}^n$.** In the next two Theorems we suppose $m, n, n-m \in \mathbb{N}^+$ and we identify 

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \text{ with } \sum_{i=1}^{m} x_i e_i + \sum_{j=1}^{n-m} y_j e_{m+j} \in \mathbb{R}^n;$$

equivalently, we identify 

$$z \in \mathbb{R}^n \text{ with } (p(z), q(z)) \in \mathbb{R}^m \times \mathbb{R}^{n-m}.$$ 

**Theorem 3.7.** Suppose $m, n, n-m \in \mathbb{N}^+$. Then 

$$\mathcal{M}_n = a(\mathcal{M}_m, \mathcal{M}_{n-m})$$

and 

$$|| \cdot ||_n = || \cdot ||_m \otimes || \cdot ||_{n-m}.$$ 

**Proof.** Induct on $n$ making use of Theorem 3.2. 

$\square$
Theorem 3.8. Suppose \( u \in S_n^+ \). Then

(i) for each \( y \in \mathbb{R}^{n-m}, \mathbb{R}^m \ni x \mapsto u(x,y) \) is \( \mathcal{M}_m \)-elementary;
(ii) \( \mathbb{R}^{n-m} \ni y \mapsto I_m(\mathbb{R}^m \ni x \mapsto u(x,y)) \) is \( \mathcal{M}_{n-m} \)-elementary;
(iii) for each \( x \in \mathbb{R}^m, \mathbb{R}^{n-m} \ni y \mapsto u(x,y) \) is \( \mathcal{M}_{n-m} \)-elementary;
(iv) \( \mathbb{R}^m \ni x \mapsto I_{n-m}(\mathbb{R}^{n-m} \ni y \mapsto u(x,y)) \) is \( \mathcal{M}_m \)-elementary;
(v)

\[
I_{n-m}(\mathbb{R}^{n-m} \ni y \mapsto I_m(\mathbb{R}^m \ni x \mapsto u(x,y))) = I_m(\mathbb{R}^m \ni x \mapsto I_{n-m}(\mathbb{R}^{n-m} \ni y \mapsto u(x,y))).
\]

Moreover, if \( Y \) is a vector space, \( u \in S_n(Y) \) and \( u \) is \( \| \cdot \|_n \)-integrable. Then

(i) \( \mathbb{R}^m \ni x \mapsto u(x,y) \) is \( \| \cdot \|_m \)-integrable for each \( y \in \mathbb{R}^{n-m} \);
(ii) \( \mathbb{R}^{n-m} \ni y \mapsto I_m(\mathbb{R}^m \ni x \mapsto u(x,y)) \) is \( \| \cdot \|_{n-m} \)-integrable;
(iii) \( \mathbb{R}^{n-m} \ni y \mapsto u(x,y) \) is \( \| \cdot \|_{n-m} \)-integrable for each \( x \in \mathbb{R}^m \);
(iv) \( \mathbb{R}^m \ni x \mapsto I_{n-m}(\mathbb{R}^{n-m} \ni y \mapsto u(x,y)) \) is \( \| \cdot \|_m \)-integrable;
(v)

\[
I_{n-m}(Y \ni y \mapsto I_m(\mathbb{R}^m \ni x \mapsto u(x,y))) = I_m(\mathbb{R}^m \ni x \mapsto I_{n-m}(Y \ni y \mapsto u(x,y))).
\]

Proof. This follows directly from the preceding Theorem and Theorem 3.3. \( \square \)

3.3. More useful facts about multirectangles.

Theorem 3.9. Suppose \( n > 1; \mathcal{C} \) is a nonempty finite subfamily of \( \mathcal{M}_n \); for each \( j \in \{1, \ldots, n\} \),

\[
\mathcal{D}_j = B(\{q_j|C \cap p_j^{-1}[w]| : C \in \mathcal{C} \text{ and } w \in \mathbb{R}^{n-1}\});
\]

\( \Gamma \) is the set of functions

\[
\gamma : \{1, \ldots, n\} \to \prod_{j=1}^n \mathcal{D}_j
\]
such that \( \gamma(j) \) is a connected component of some member of \( \mathcal{D}_j \) whenever \( j \in \{1, \ldots, n\} \); and

\[
\mathcal{F} = \left\{ \prod_{j=1}^n \gamma(j) : \gamma \Gamma \right\}.
\]

Then \( \mathcal{F} \) is a finite disjointed family of rectangles and

\[
C = \cup \{R : R \in \mathcal{F} \text{ and } F \subset C\} \quad \text{whenever } C \in \mathcal{C}.
\]

Proof. From our previous work it follows that \( \mathcal{D}_j \) is a finite disjointed family of multintervals for each \( j \in \{1, \ldots, n\} \). From the Product Decomposition and induction on \( n \) we find that

\[
C = \prod_{j=1}^m M_j : (M_1, \ldots, M_n) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \text{ and } \prod_{j=1}^n M_j \subset C \quad \text{whenever } C \in \mathcal{C}.
\]

To complete the proof one only has to observe that a multinterval is the finite disjoint union of its connected components each of which is an interval. \( \square \)
Corollary 3.1. Suppose $S$ is a finite family of $M_n$-elementary functions with values in $Z$ where either $Z = [0, \infty]$ or $Z$ is a vector space. Then there are a finite disjointed family $R$ of rectangles and, for each $s \in S$, a function $\sigma_s : R \to Z$ such that
\[
s = \sum_{R \in R} \sigma_s(R)1_R \quad \text{whenever } s \in S.
\]

Proposition 3.1. The interior, closure and boundary of a multirectangle in $\mathbb{R}^n$ are is a multirectangle in $\mathbb{R}^n$.

Proof. Suppose $\mathcal{F}$ is a finite family of rectangles in $\mathbb{R}^n$ and $M = \bigcup \mathcal{F}$. Since the closure of the union of a finite family of closed sets in a topological space is the union of the closures (Proof?) and since the closure of a rectangle is a rectangle we find that
\[
cl M = \bigcup \{cl R : R \in \mathcal{F}\} \in M_n.
\]
This in turn implies
\[
bdry M = (cl M) \cap (cl (\mathbb{R}^n \sim M)) \in M_n
\]
and
\[
int M = (cl M) \sim (int M) \in M_n.
\]

Proposition 3.2. Suppose $M$ is a multirectangle in $\mathbb{R}^n$. Then
\[
|| bdry M ||_n = 0.
\]

Proof. Let $\mathcal{F}$ be a finite family of rectangles in $\mathbb{R}^n$ such that $M = \bigcup \mathcal{F}$. Now any boundary point of $M$ cannot be interior to any $R \in \mathcal{F}$ which implies that $bdry F \subset \bigcup \{bdry R : R \in \mathcal{F}\}$. Thus
\[
|| bdry F ||_n \leq \sum_{R \in \mathcal{F}} || bdry R ||_n = 0
\]
since $|| bdry R ||_n = 0$ for any rectangle in $\mathbb{R}^n$. \qed

Definition 3.3. We say a finite family $\mathcal{F}$ of multirectangles in nonoverlapping if
\[
(int M) \cap (int N) = \emptyset \quad \text{whenever } M, N \in \mathcal{F} \text{ and } M \neq N.
\]

Proposition 3.3. Suppose $\mathcal{F}$ is a finite nonoverlapping family of multirectangles. Then
\[
|| \bigcup \mathcal{F} ||_n = \sum_{M \in \mathcal{F}} || M ||_n.
\]

Proof. We have
\[
\sum_{M \in \mathcal{F}} || M ||_n = \sum_{M \in \mathcal{F}} || int M ||_n = \sum_{M \in \mathcal{F}} || \bigcup \{int M : M \in \mathcal{F}\} ||_n \leq || \bigcup \mathcal{F} ||_n \leq \sum_{M \in \mathcal{F}} || F ||_n.
\]
\qed