1. Algebras and σ-algebras of sets and the integration of simple functions.

Let X be a nonempty set.

Definition 1.1. Suppose $\mathcal{S}$ is a family of subsets of X. We let

$$\begin{aligned}
\mathbf{u}(\mathcal{S}) &= \{\cup\mathcal{F} : \mathcal{F} \text{ is a finite subfamily of } \mathcal{S}\};\\
\mathbf{d}(\mathcal{S}) &= \{\cup\mathcal{F} : \mathcal{F} \text{ is a finite disjointed subfamily of } \mathcal{S}\};\\
\mathbf{i}(\mathcal{S}) &= \{\cap\mathcal{F} : \mathcal{F} \text{ is a nonempty finite subfamily of } \mathcal{S}\};\\
\mathbf{c}(\mathcal{S}) &= \{X \sim A : A \in \mathcal{S}\}.
\end{aligned}$$

We also set

$$\begin{aligned}
\mathbf{u}_\sigma(\mathcal{S}) &= \{\cup\mathcal{F} : \mathcal{F} \text{ is a countable subfamily of } \mathcal{S}\};\\
\mathbf{d}_\sigma(\mathcal{S}) &= \{\cup\mathcal{F} : \mathcal{F} \text{ is a countable disjointed subfamily of } \mathcal{S}\};\\
\mathbf{i}_\sigma(\mathcal{S}) &= \{\cap\mathcal{F} : \mathcal{F} \text{ is a nonempty countable subfamily of } \mathcal{S}\}.
\end{aligned}$$

Since $\emptyset$ is a finite subfamily of $\mathcal{S}$ we find that $\emptyset = \cup\emptyset \in \mathbf{u}(\mathcal{S})$; since $\emptyset$ is a countable subfamily of $\mathcal{S}$ we find that $\emptyset = \cup\emptyset \in \mathbf{u}_\sigma(\mathcal{S})$.

Definition 1.2. Suppose $\mathcal{A}$ is a family of subsets of X. We say $\mathcal{A}$ is an **algebra of subsets of** X if $\mathbf{u}(\mathcal{S}) \subset \mathcal{S}$ and $\mathbf{c}(\mathcal{S}) \subset \mathcal{S}$. We say $\mathcal{A}$ is a **σ-algebra of subsets of** X if $\mathbf{u}_\sigma(\mathcal{S}) \subset \mathcal{S}$ and $\mathbf{c}(\mathcal{S}) \subset \mathcal{S}$.

Proposition 1.1. Suppose $\mathcal{A}$ is a family of subsets of X and $\mathbf{c}(\mathcal{A}) \subset \mathcal{A}$. Then $\mathcal{A}$ is an algebra of subsets of X if and only if $\emptyset \in \mathcal{A}$ and $\mathbf{i}(\mathcal{A}) \subset \mathcal{A}$ and $\mathcal{A}$ is a σ-algebra of subsets of X if and only if $\emptyset \in \mathcal{A}$ and $\mathbf{i}_\sigma(\mathcal{A}) \subset \mathcal{A}$.

Proof. This follows directly from the DeMorgan Laws. □

Definition 1.3. Suppose $\mathcal{S}$ is a family of subsets of X. Note that 2^X is an algebra of subsets of X and $\mathcal{S} \subset 2^X$ so that

$$\{\mathcal{A} : \mathcal{A} \text{ is an algebra of subsets of } X \text{ and } \mathcal{S} \subset \mathcal{A}\} \neq \emptyset$$

so

$$\mathbf{a}(\mathcal{S}) = \cap\{\mathcal{A} : \mathcal{A} \text{ is an algebra of subsets of } X \text{ and } \mathcal{S} \subset \mathcal{A}\}$$

is well defined.

Also, as 2^X is a σ-algebra of subsets of X and that $\mathcal{S} \subset 2^X$ so that

$$\{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } X \text{ and } \mathcal{S} \subset \mathcal{A}\} \neq \emptyset$$

so

$$\mathbf{a}_\sigma(\mathcal{S}) = \cap\{\mathcal{A} : \mathcal{A} \text{ is an } \sigma\text{-algebra of subsets of } X \text{ and } \mathcal{S} \subset \mathcal{A}\}$$

is well defined.

Theorem 1.1. Suppose $\mathcal{S}$ is a family of subsets of X. Then

(i) $\mathbf{a}(\mathcal{S})$ is an algebra of subsets of X;
(ii) $\mathcal{S} \subset \mathbf{a}(\mathcal{S})$;
(iii) if $\mathcal{A}$ is an algebra of subsets of X and $\mathcal{S} \subset \mathcal{A}$ then $\mathbf{a}(\mathcal{S}) \subset \mathcal{A}$.

Moreover,

(iv) $\mathbf{a}_\sigma(\mathcal{S})$ is a σ-algebra of subsets of X;

(v) $\mathcal{S} \subset \mathbf{a}(\mathcal{S})$;
(vi) if $\mathcal{A}$ is a σ- algebra of subsets of X and $\mathcal{S} \subset \mathcal{A}$ then $\mathbf{a}(\mathcal{S}) \subset \mathcal{A}$.

Proof. This is straightforward once you get used to the definitions.

Suppose $\mathcal{F} \subset \mathbf{a}(\mathcal{S})$ and $\mathcal{F}$ is finite. If $\mathcal{A}$ is an algebra of subsets of X and $\mathcal{S} \subset \mathcal{A}$ then $\mathcal{F} \subset \mathcal{A}$ so $\cup\mathcal{F} \in \mathbf{u}(\mathcal{A}) \subset \mathcal{A}$ and, if $\mathcal{F}$ is nonempty, $\cap\mathcal{F} \in \mathbf{i}(\mathcal{A}) \subset \mathcal{A}$. If $A \in \mathbf{a}(\mathcal{S})$ then $A \in \mathcal{A}$ so $X \sim A \in \mathcal{A}$. So (i) holds.

Suppose $A \in \mathcal{S}$. If $\mathcal{A}$ is an algebra of subsets of X and $\mathcal{S} \subset \mathcal{A}$ then $A \in \mathcal{A}$. Thus $A \in \mathbf{a}(\mathcal{S})$ so (ii) holds.

(iii) is a direct consequence of the definition of $\mathbf{a}(\mathcal{S})$.

Similar arguments prove (iii)-(iv). □

Definition 1.4. So algebras and σ-algebras of subsets containing a given family of subsets exist. One calls $\mathbf{a}(\mathcal{S})$ the **algebra of subsets of X generated by $\mathcal{S}$**; by (iii) it is the *smallest* algebra of subsets of X containing $\mathcal{S}$ and one calls $\mathbf{a}_\sigma(\mathcal{S})$ the **σ-algebra of subsets of X generated by $\mathcal{S}$**; by (vi) it is the *smallest* σ-algebra of subsets of X containing $\mathcal{S}$.

1.1. **Some results on algebras of subsets of X.**

Proposition 1.2. Suppose $\mathcal{S}$ is a family of subsets of X. Then

$$\mathbf{a}(\mathcal{S}) = \cup\{\mathbf{a}(\mathcal{F}) : \mathcal{F} \text{ is a finite subfamily of } \mathcal{S}\}.$$

Proof. Exercise for the reader. □

The next Proposition will be very useful to us. To prove it we will make use of the following Lemma.

Lemma 1.1. Suppose $\mathcal{R}$ is a family of subsets of X. Then
(i) $\mathbf{u}(\mathcal{R}) \subset \mathcal{R}$ provided $\emptyset \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$;
(ii) $\mathbf{i}(\mathcal{R}) \subset \mathcal{R}$ provided $A \cap B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$.

Proof. Suppose $\emptyset \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$. Let $\mathcal{M}$ be the set of $n \in \mathbb{N}$ such that $\cup\mathcal{F} \in \mathcal{M}$ if $\mathcal{F} \subset \mathcal{R}$ and $\mathbf{card}\,\mathcal{F} = n$. We will show by induction on $n \in \mathcal{M}$ that $\mathcal{M} = \mathbb{N}$ thereby proving (i). It is trivial that $\{0, 1\} \in \mathcal{M}$. Suppose $n \in \mathcal{M}$, $\mathcal{F} \subset \mathcal{R}$ and $\mathbf{card}\,\mathcal{F} = n + 1$. Choose $A \in \mathcal{F}$, let $\mathcal{G} = \mathcal{F} \sim \{A\}$ and let $B = \cup\mathcal{G}$. Then $\cup\mathcal{F} = A \cup (\cup\mathcal{G}) = A \cup B \in \mathcal{R}$ since $\mathbf{card}\,\mathcal{G} = n \in \mathcal{M}$. So $\mathcal{M} = \mathbb{N}$.

Suppose $A \cup B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$. If $\mathcal{R}$ is empty then (ii) holds trivially so suppose $\mathcal{R} \neq \emptyset$. Let $\mathcal{N}$ be the set of $n \in \mathbb{N}^+$ such that $\cap\mathcal{F} \in \mathcal{N}$ if $\mathcal{F} \subset \mathcal{R}$ and $\mathbf{card}\,\mathcal{F} = n$. We will show by induction on $n \in \mathcal{N}$ that $\mathcal{N} = \mathbb{N}$ thereby proving (ii). It is trivial that $1 \in \mathcal{N}$. Suppose $n \in \mathcal{N}$, $\mathcal{F} \subset \mathcal{R}$ and $\mathbf{card}\,\mathcal{F} = n + 1$. Choose $A \in \mathcal{F}$, let $\mathcal{G} = \mathcal{F} \sim \{A\}$ and let $B = \cap\mathcal{G}$. Then $\cap\mathcal{F} = A \cap (\cap\mathcal{G}) = A \cap B \in \mathcal{R}$ since $\mathbf{card}\,\mathcal{G} = n \in \mathcal{N}$. So $\mathcal{N} = \mathbb{N}^+$.

□

Proposition 1.3. Suppose $\mathcal{S}$ is a family of subsets of X. Then
(i) $\mathbf{u}(\mathbf{u}(\mathcal{S})) = \mathbf{u}(\mathcal{S})$;
(ii) if $\mathbf{i}(\mathcal{S}) \subset \mathcal{S}$ then $\mathbf{i}(\mathbf{u}(\mathcal{S})) \subset \mathbf{u}(\mathcal{S})$;
(iii) if $\mathbf{i}(\mathcal{S}) \subset \mathcal{S}$ then $\mathbf{i}(\mathbf{d}(\mathcal{S})) \subset \mathbf{d}(\mathcal{S})$;
(iv) If $\mathbf{i}(\mathcal{S}) \subset \mathcal{S}$ and $\mathbf{c}(\mathcal{S}) \subset \mathbf{u}(\mathcal{S})$ then $\mathbf{a}(\mathcal{S}) = \mathbf{u}(\mathcal{S})$;
(v) If $\mathbf{i}(\mathcal{S}) \subset \mathcal{S}$ and $\mathbf{c}(\mathcal{S}) \subset \mathbf{d}(\mathcal{S})$ then $\mathbf{a}(\mathcal{S}) = \mathbf{d}(\mathcal{S})$;

Proof. Suppose $A, B \in \mathbf{u}(\mathcal{S})$. There are finite subfamilies $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{S}$ such that $A = \cup\mathcal{F}$ and $B = \cup\mathcal{G}$.

Since $A \cup B = \cup(\mathcal{F} \cup \mathcal{G}) \in \mathbf{u}(\mathcal{S})$ and $\emptyset \in \mathbf{u}(\mathcal{S})$ we infer from (i) of the preceding Lemma with $\mathcal{R}$ there equal $\mathbf{u}(\mathcal{S})$ that (i) holds.

For the remainder of the proof we suppose $\mathbf{i}(\mathcal{S}) \subset \mathcal{S}$.

We have $A \cap B = \cup\{F \cap G : (F, G) \in \mathcal{F} \times \mathcal{G}\} \subset \mathbf{u}(\mathcal{S})$ so we infer from (ii) of the preceding Lemma that with $\mathcal{R}$ there equal $\mathbf{u}(\mathcal{S})$ that (ii) holds.

If $A, B \in \mathbf{d}(\mathcal{S})$ we can choose $\mathcal{F}$ and $\mathcal{G}$ to be disjointed in which case $\cup(\{F \cap G : (F, G) \in \mathcal{F} \times \mathcal{G}\} \subset \mathbf{i}(\mathcal{S})$ so we infer from (ii) of the preceding Lemma that with $\mathcal{R}$ there equal $\mathbf{i}(\mathcal{S})$ that (iii) holds.

Suppose $\mathcal{H}$ is nonempty finite subfamily of $\mathcal{S}$ and $C = \cup\mathcal{H}$. Then

$$X \sim C = \cap\{X \sim D : D \in \mathcal{H}\} \in \begin{cases} \mathbf{i}(\mathbf{u}(\mathcal{S})) & \text{if } \mathbf{c}(\mathcal{S}) \subset \mathbf{u}(\mathcal{S}), \\ \mathbf{i}(\mathbf{d}(\mathcal{S})) & \text{if } \mathbf{c}(\mathcal{S}) \subset \mathbf{d}(\mathcal{S}). \end{cases}$$

If $\mathbf{c}(\mathcal{S}) \subset \mathbf{u}(\mathcal{S})$ then $\mathbf{i}(\mathbf{u}(\mathcal{S})) \subset \mathbf{u}(\mathcal{S})$ by (ii) and (iv) holds. If $\mathbf{c}(\mathcal{S}) \subset \mathbf{d}(\mathcal{S})$ then $\mathbf{i}(\mathbf{d}(\mathcal{S})) \subset \mathbf{d}(\mathcal{S})$ by (iii) and (v) holds. (ii) and (iv) holds.

□

1.2. In 5.1 we provide a construction of $\mathbf{a}(\mathcal{S})$ when $\mathcal{S}$ is a finite subfamily of X.

1.3. Additive nonegative extended real valued functions on an algebra of subsets of X.

Definition 1.5. Suppose $\mathcal{A}$ is a algebra of subsets of X. We say a nonegative extended real valued function α on $\mathcal{A}$ is **additive** if

$$\alpha(\cup\mathcal{S}) = \sum_{S \in \mathcal{S}} \alpha(S) \quad \text{whenever } \mathcal{S} \text{ is a finite disjointed subfamily of } \mathcal{A}.$$

Proposition 1.4. Suppose $\mathcal{A}$ is a algebra of subsets of X and $\alpha : \mathcal{A} \to [0, \infty]$. The following statements are equivalent:

(i) α is a nonegative extended real valued function $\mathcal{A}$;

(ii) $\alpha(\emptyset) = 0$ and
$$\alpha(A \cup B) + \alpha(A \cap B) = \alpha(A) + \alpha(B) \quad \text{whenever } A, B \in \mathcal{A};$$

(iii) $\alpha(\emptyset) = 0$ and
$$\alpha(A \cup B) = \alpha(A) + \alpha(B) \quad \text{whenever } A, B \in \mathcal{A} \text{ and } A \cap B = \emptyset;$$

Proof. Suppose (i) holds. The $\emptyset$ is a finite subfamily of $\mathcal{A}$ so $\alpha(\emptyset) = \alpha(\cup\emptyset) = \sum_{S \in \emptyset} \alpha(S) = 0$. Suppose $A, B \in \mathcal{A}$. Since $\{A \sim B, A \cap B, B \sim A\}$, $\{A \sim B, A \cap B\}$ and $\{B \sim A, A \cap B\}$ are subfamilies of $\mathcal{A}$ which are partitions of $A \cup B$, A and B, respectively, we have

$$\alpha(A \cup B) + \alpha(A \cap B) = \alpha(A \sim B) + \alpha(A \cap B) + \alpha(B \sim A) + \alpha(A \cap B)$$

and

$$\alpha(A) + \alpha(B) = \alpha(A \sim B) + \alpha(A \cap B) + \alpha(B \sim A) + \alpha(A \cap B)$$

so (i) implies (ii).

Suppose (ii) holds, $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$. Since $\alpha(A \cap B) = 0$ we infer that (ii) implies (iii).

Suppose (iii) holds, $\mathcal{F}$ is a finite disjointed subfamily of $\mathcal{A}$, $\alpha(\cup\mathcal{F}) = \sum_{F\in\mathcal{F}}\alpha(F)$, $A \in \mathcal{A}$ and $A \cap (\cup\mathcal{F}) = \emptyset$. Then

$$\alpha(A \cup (\cup\mathcal{F})) = \alpha(A) + \alpha(\cup\mathcal{F}) = \sum_{F\in\{A\}\cup\mathcal{F}} \alpha(F).$$

So, by induction on $\mathbf{card}\,\mathcal{F}$, (iii) implies (i). □

Proposition 1.5 (The Inclusion-Exclusion Principle.)**.** Suppose $\mathcal{A}$ is an algebra of subsets of X and α is an additive nonnegative extended real valued function on $\mathcal{A}$. For $N \in \mathbb{N}^+$ and $S \in \mathcal{A}^N$ we set

$$\mathbf{s}_o(N) = \{\sigma : \sigma \subset \mathbb{J}(1,N) \text{ and } \mathbf{card}\,\sigma \text{ is odd}\};$$
$$\mathbf{s}_e(N) = \{\sigma : \sigma \subset \mathbb{J}(1,N),\ \sigma \neq \emptyset \text{ and } \mathbf{card}\,\sigma \text{ is even}\};$$

and

$$S_\sigma = \cap\{S(i) : i \in \sigma\} \quad \text{for } \sigma \in \mathbf{S}_o \cup \mathbf{S}_e.$$

For any $N \in \mathbb{N}^+$ and $S \in \mathcal{A}^N$ we have

$$\alpha\left(\cup_{i=1}^N S(i)\right) + \sum_{\sigma\in\mathbf{s}_e(N)} \alpha\left(S_\sigma\right) = \sum_{\sigma\in\mathbf{s}_o(N)} \alpha\left(S_\sigma\right). \tag{1}$$

Proof. Let $\mathcal{N}$ be the set of $N \in \mathbb{N}^+$ for which the Proposition holds. That $1 \in \mathcal{N}$ is trivial.

Suppose $N \in \mathbb{N}$, $N > 1$, $N-1 \in \mathcal{N}$ and $S \in \mathcal{A}^N$. We may assume that $\alpha(S_i) < \infty$ for $i \in \mathbb{J}(1,N)$ since otherwise both sides of (1) equal ∞. Let $R \in \mathcal{A}^{N-1}$ be such that $R_i = S_i \cap S_N$ for $i \in \mathbb{J}(1,N-1)$.

By ?? we have

$$\begin{aligned}
\alpha(\cup_{i=1}^N S_i) &= \alpha(\cup^{N-1} S_i) + \alpha(S_N) - \alpha(\cup_{i=1}^{N-1} R_i) \\
&= \sum_{\sigma\in\mathbf{s}_o(N),\ \sigma\subset\mathbb{J}(1,N-1)} \alpha(S_\sigma) - \sum_{\sigma\in\mathbf{s}_e(N),\ \sigma\subset\mathbb{J}(1,N-1)} \alpha(S_\sigma) \\
&\quad + \alpha(S_N) \\
&\quad - \sum_{\sigma\in\mathbf{s}_o(N-1),\ \sigma\subset\mathbb{J}(1,N-1)} \alpha(R_\sigma) + \sum_{\sigma\in\mathbf{s}_e(N-1),\ \sigma\subset\mathbb{J}(1,N-1)} \alpha(R_\sigma)
\end{aligned}$$

For $\sigma \in \mathbf{s}_o(N-1) \cup \mathbf{s}_e(N-1)$ we set $\tilde\sigma = \sigma \cup \{N\}$; obviously,

$$\sigma \in \mathbf{s}_o(N-1) \ \Leftrightarrow\ \tilde\sigma \in \mathbf{s}_e(N), \qquad \sigma \in \mathbf{s}_e(N-1) \ \Leftrightarrow\ \tilde\sigma \in \mathbf{s}_o(N)$$

and $R_\sigma = S_{\tilde\sigma}$for $\sigma \in \mathbf{s}_o(N-1) \cup \mathbf{s}_e(N-1)$. □

1.4. Multiintervals: A basic example.

Definition 1.6. Let $\mathbb{Int}$ be the family of intervals in $\mathbb{R}$ and let

$$\mathbb{MultiInt} = \mathbf{d}(\mathbb{Int}).$$

The members of $\mathbb{MultiInt}$ are called **multiintervals**.

Definition 1.7. Whenever $I \in \mathbb{Int}$ we let

$$I^- = \{y \in \mathbb{R} : y < x \text{ whenever } x \in I\} \quad \text{and} \quad I^+ = \{y \in \mathbb{R} : x < y \text{ whenever } x \in I\}.$$

Note that if $I = \emptyset$ then $I^- = \mathbb{R} = I^+$.

Proposition 1.6. Suppose $I \in \mathbb{Int} \sim \{\emptyset\}$. Then $I^-, I^+ \in \mathbb{Int}$, $\{I^-, I, I^+\}$ is a partition of $\mathbb{R}$ and $\mathbb{R} \sim I = I^- \cup I^+$.

Proof. We leave this as a straightforward exercise for the reader. □

Proposition 1.7. $\mathbb{MultiInt}$ is an algebra of subsets of $\mathbb{R}$.

Proof. Suppose $I, J \in \mathbb{Int}$. Then $I \sim J = (I \cap J^-) \cup (I \cap J^+)$ so $\mathbf{c}(\mathbb{Int}) \subset \mathbf{d}(\mathbb{Int})$. Obviously, $\mathbf{i}(\mathbb{Int}) \subset \mathbb{Int}$. So the Proposition follows from Proposition 1.3(v). □

Proposition 1.8. A subset of $\mathbb{R}$ is a multiinterval if and only if it has finitely many connected components.

Proof. Suppose $A \subset \mathbb{R}$ and $\mathcal{C}$ is the family of connected components of A. If $\mathcal{C}$ is finite then A is a multiinterval because $A = \cup\mathcal{C}$. If, on the other hand, $A = \cup\mathcal{F}$ where $\mathcal{F}$ is a finite subfamily of $\mathbb{Int}$ then

$$F = \{(I, C) \in \mathcal{F} \times \mathcal{C} : I \subset C\}$$

is a function with a finite domain $\mathcal{I}$ whose range equals $\mathcal{C}$; thus $\mathcal{C}$ is finite. □

Lemma 1.2. Suppose $\mathcal{J}$ is a finite disjointed family of nonempty intervals and $I = \cup\mathcal{J} \in \mathbb{Int}$. Then

$$\mathbf{diam}\, I = \sum_{J \in \mathcal{J}} \mathbf{diam}\, J.$$

Proof. We prove this by induction on the number of members of $\mathcal{J}$. The assertion holds trivially if $\mathcal{J}$ has no members or one member so suppose $\mathcal{J}$ has at least two members and that the Lemma holds for families having fewer members than $\mathcal{J}$. Let $l = \{(J_1, J_2) : J_1, J_2 \in \mathcal{J} \text{ and } J_1 \subset J_2^-\}$ and note that l linearly orders $\mathcal{J}$. Let J be the unique l-largest element of $\mathcal{J}$ and let $\mathcal{K} = \mathcal{J} \sim \{J\}$. We leave it as a simple exercise for the reader to verify that

$$\cup\mathcal{K} \text{ is an interval}; \quad \sup \cup\mathcal{K} = \inf J; \quad \mathbf{diam}\, I = \mathbf{diam} \cup \mathcal{K} + \mathbf{diam}\, J.$$

The Lemma now follows. □

Theorem 1.2. There is one and only one additive nonnegative extended real valued function

$$|| \cdot ||$$

on the algebra of multiintervals such that

$$||I|| = \mathbf{diam}\, I \quad \text{whenever } I \text{ is a nonempty interval in } \mathbb{R}.$$

Proof. Let $||\emptyset|| = 0$ and, given a nonempty multiinterval M, we let $||M||$ be the sum of the diameters of its connected components. Suppose $M_i,\ i = 1, 2$ are multiintervals and $M_1 \cap M_2 = \emptyset$. Let $\mathcal{I}_i,\ i = 1, 2$, be the families of connected components of $M_i,\ i = 1, 2$, respectively. Let $\mathcal{J}$ be the family of connected components of $M_1 \cup M_2$. Then, and this is the main point, (i) each member of $\mathcal{I}_1 \cup \mathcal{I}_2$ is contained in a unique member of $\mathcal{J}$ and (ii) each member of $\mathcal{J}$ is the disjoint union of the members of $\mathcal{I}_1 \cup \mathcal{I}_2$ it contains. Making use of Lemma 1.2 we find that

$$\begin{aligned}
||M_1 \cup M_2|| &= \sum_{J \in \mathcal{J}} \mathbf{diam}\, J \\
&= \sum_{J \in \mathcal{J}} \Big(\sum_{I_1 \in \mathcal{I}_1,\ I_1 \subset J} \mathbf{diam}\, I_1 + \sum_{I_2 \in \mathcal{I}_2,\ I_2 \subset J} \mathbf{diam}\, I_1 \Big) \\
&= \sum_{I_1 \in \mathcal{I}_1} \mathbf{diam}\, I_1 + \sum_{I_2 \in \mathcal{I}_2} \mathbf{diam}\, I_2 \\
&= ||M_1|| + ||M_2||.
\end{aligned}$$

□

Remark 1.1. $||\cdot||$ has the following desirable property: If M is a nondecreasing sequence in $\mathbb{MultiInt}$ and $\cup_{\nu=0}^{\infty} \in M_\nu \in \mathbb{MultiInt}$ then

$$|| \cup_{\nu=0}^{\infty} M_\nu|| = \sup_\nu ||M_\nu||.$$

This is ?? in case n there equals 1.

Example 1.1. Let $X = (0,1)$, let $\mathcal{A} = \{M \in \mathbb{MultiInt} : M \subset X\}$ and let $\alpha : \mathcal{A} \to [0,1]$ be such that

$$\alpha(A) = \begin{cases} ||A|| & \text{if } A \in \mathcal{A} \text{ and } \inf A > 0, \\ ||A|| + 1 & \text{if } A \in \mathcal{A} \text{ and } \inf A = 0. \end{cases}$$

Suppose $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$. If $\inf A = 0$ then $(0,\epsilon) \subset A$ for some $\epsilon \in (0,1)$ so $\inf B > 0$; by the same argument we find that if $\inf B = 0$ then $\inf A > 0$. So exactly one of the following holds:

(i) $\inf A = 0$ and $\inf B > 0$;
(ii) $\inf A > 0$ and $\inf B = 0$;
(iii) $\inf A > 0$ and $\inf B > 0$.

If either (i) or (ii) holds then $\inf(A \cup B) = 0$ so

$$\alpha(A \cup B) = ||A \cup B|| + 1 = ||A|| + ||B|| + 1 = \alpha(A) + \alpha(B).$$

If (iii) holds then $\inf(A \cup B) > 0$ and so

$$\alpha(A \cup B) = ||A \cup B|| = ||A|| + ||B|| = \alpha(A) + \alpha(B).$$

Thus α is an additive nonnegative extended real valued function on the algebra $\mathcal{A}$ of subsets of X. Now let ϵ be a decreasing sequence in $(0,1)$ such that $\inf_\nu \epsilon_\nu = 0$ and, for each $\nu \in \mathbb{N}$, let $M_\nu = (\epsilon_\nu, 1) \in \mathcal{A}$. Then $\mathbb{N} \ni M_\nu$ is an increasing sequence in $\mathcal{A}$ and

$$\sup_\nu \alpha(M_\nu) = \sup_\nu (1 - \epsilon_\nu) = 1 < 1 + 1 = \alpha(0,1) = \alpha(\cup_\nu M_\nu).$$

2. Integration of simple functions.

Suppose $\mathcal{A}$ is an algebra of subsets of X and α is an additive nonnegative extended real value function on $\mathcal{A}$.

Definition 2.1. We let

$$\mathcal{S}_\alpha(\mathcal{A})$$

be the set of those $s \in \mathbb{R}^X$ such that

$$\textbf{rng}\, s \text{ is finite}; \quad s^{-1}[\{y\}] \in \mathcal{A} \text{ for } y \in \mathbb{R}; \quad \sum_{y \in \mathbb{R}} |y|\, \alpha(s^{-1}[\{y\}]) < \infty.$$

We define

$$I_\alpha : \mathcal{S}_\alpha(\mathcal{A}) \to \mathbb{R}$$

by setting

$$I_\alpha(s) = \sum_{y \in \mathbb{R}} y\, \alpha(s^{-1}[\{y\}]) \quad \text{for } s \in \mathcal{S}_\alpha(\mathcal{A}).$$

Remark 2.1. For *any* $s \in \mathbb{R}^X$ we have

$$s = \sum_{y \in \mathbb{R}} y\, 1_{s^{-1}[\{y\}]} = \sum_{y \in \mathbf{rng}\, s} y\, 1_{s^{-1}[\{y\}]}. \tag{2}$$

Lemma 2.1. Suppose

(i) I is a finite set, $\sigma : I \to \mathbb{R}$ and $S : I \to \mathcal{A}$;
(ii) $\mathbf{rng}\, S$ is disjointed;
(iii) $\sum_{i \in I} |\sigma(i)| \alpha(S(i)) < \infty$;
(iv) $s \in \mathbb{R}^X$ and $s = \sum_{i \in I} \sigma(i) 1_{S(i)}$.

Then $s \in \mathcal{S}_\alpha(\mathcal{A})$ and

$$I_\alpha(s) = \sum_{i \in I} \sigma(i) \alpha(S(i)).$$

Proof. For each $y \in \mathbf{rng}\, s$ we let $J(y) = \{i \in I : s^{-1}[\{y\}] \cap S(i) \neq \emptyset\}$. Evidently,

$$I \sim \cup_{y \in \mathbf{rng}\, s} J(y) = \{i \in I : S(i) = \emptyset\}.$$

I claim that

$$s^{-1}[\{y\}] = \cup_{i \in J(y)} S(i) \in \mathcal{A} \quad \text{and} \quad \{J(y) : y \in \mathbf{rng}\, s\} \text{ is disjointed.} \tag{3}$$

Indeed, suppose $y \in \mathbf{rng}\, s$. Then

$$y\, 1_{s^{-1}[\{y\}]} = 1_{s^{-1}[\{y\}]}\, s = \sum_{i \in I} \sigma(i) 1_{s^{-1}[\{y\}] \cap S(i)} = \sum_{i \in J(y)} \sigma(i) 1_{s^{-1}[\{y\}] \cap S(i)}.$$

So $\sigma(i) = y$ for $i \in J(y)$ and $s^{-1}[\{y\}] = \cup_{i \in J(y)} s^{-1}[\{y\}] \cap S(i)$. If $i \in J(y)$, $z \in \mathbf{rng}\, s$ and $s^{-1}[\{z\}] \cap S(i) \neq \emptyset$ then $i \in J(z)$ and $\sigma(i) = z$ so $z = y$. So (3) holds and $s \in \mathcal{S}_\alpha(\mathcal{A})$.

Moreover,

$$\begin{aligned} I_\alpha(s) &= \sum_{y \in \mathbb{R}} y\, \alpha(s^{-1}[\{y\}]) \\ &= \sum_{y \in \mathbf{rng}\, s} y\, \alpha(s^{-1}[\{y\}]) \\ &= \sum_{y \in \mathbf{rng}\, s} y \left(\sum_{i \in J(y)} \alpha(S(i)) \right) \\ &= \sum_{y \in \mathbf{rng}\, s} \left(\sum_{i \in J(y)} \sigma(i) \alpha(S(i)) \right) \\ &= \sum_{i \in I,\ S(i) \neq \emptyset} \sigma(i) \alpha(S(i)) \\ &= \sum_{i \in I} \sigma(i) \alpha(S(i)) \end{aligned}$$

□

Proposition 2.1. $\mathcal{S}_\alpha(\mathcal{A})$ is a linear subspace of $\mathbb{R}^X$ and I_α is linear. Moreover,

(i) if $A \in \mathcal{A}$ and $\alpha(A) < \infty$ then $1_A \in \mathcal{S}_\alpha(\mathcal{A})$ and $I_\alpha(1_A) = \alpha(A)$.
(ii) if $c \in \mathbb{R}^+$ and $s \in \mathcal{S}_\alpha(\mathcal{A})$ then $s \wedge c \in \mathcal{S}_\alpha(\mathcal{A})$;
(iii) if $s, t \in \mathcal{S}_\alpha(\mathcal{A})$ then $s \wedge t \in \mathcal{S}_\alpha(\mathcal{S})$;

(iv) if $s,t \in \mathcal{S}_\alpha(\mathcal{A})$ and $s \le t$ then $I_\alpha(s) \le I_\alpha(t)$.

Proof. (i) is obvious.

If $s \in \mathcal{S}_\alpha(\mathcal{A})$ and $c \in \mathbb{R}$ it is obvious that

$$cs \in \mathcal{S}_\alpha(\mathcal{A}) \text{ and } I_\alpha(cs) = cI_\alpha(s). \tag{4}$$

Also, since $s \wedge c = \sum_{y \in \mathbf{rng}\, s} (y \wedge c) 1_{s^{-1}[\{y\}]}$, we find with the help of Lemma 2.1 that (ii) holds.

Suppose $s,t \in \mathcal{S}_\alpha(\mathcal{A})$. For $(y,z) \in \mathbb{R} \times \mathbb{R}$ let $R(y,z) = s^{-1}[\{y\}] \cap t^{-1}[\{z\}]$. Then

$$s = \sum_{y \in \mathbf{rng}\, s} y\, 1_{s^{-1}[\{y\}]} = \sum_{y \in \mathbf{rng}\, s} y \left(\sum_{z \in \mathbf{rng}\, t} 1_{R(y,z)} \right) = \sum_{(y,z) \in \mathbf{rng}\, s \times \mathbf{rng}\, t} y\, 1_{R(y,z)}; \tag{5}$$

by a similar calculation,

$$t = \sum_{(y,z) \in \mathbf{rng}\, s \times \mathbf{rng}\, t} z\, 1_{R(y,z)}. \tag{6}$$

That $s + t \in \mathcal{S}_\alpha(\mathcal{A})$ now follows from Lemma 2.1

Applying Lemma 2.1 three times we find that

$$I_\alpha(s) = \sum_{(y,z) \in \mathbf{rng}\, s \times \mathbf{rng}\, t} y\, \alpha(R(y,z)); \tag{7}$$

$$I_\alpha(t) = \sum_{(y,z) \in \mathbf{rng}\, s \times \mathbf{rng}\, t} z\, \alpha(R(y,z)); \tag{8}$$

$$I_\alpha(s+t) = \sum_{(y,z) \in \mathbf{rng}\, s \times \mathbf{rng}\, t} (y+z)\, \alpha(R(y,z)). \tag{9}$$

It follows that $I_\alpha(s+t) = I_\alpha(s) + I_\alpha(t)$. Keeping in mind (4) we find that $\mathcal{S}_\alpha(\mathcal{A})$ is a linear subspace of $\mathbb{R}^X$ and I_α is linear.

From (5) and (6) we find that

$$s \wedge t = \sum_{(y,z) \in \mathbf{rng}\, r \times \mathbf{rng}\, t} (y \wedge z) 1_{R(y,z)}$$

so (iii) follows from Lemma 2.1.

Examining (7) and (8) we find that (iv) holds. □

2.1. Mapping.

Proposition 2.2. Suppose

(i) Y is a set and $f : X \to Y$;
(ii) $\mathcal{A}$ is an algebra of subsets of X;
(iii) $\mathcal{B} = \{Y \sim \mathbf{rng}\, f\} \cup \{B : B \subset \mathbf{rng}\, f \text{ and } f^{-1}[B] \in \mathcal{A}\}$.

Then

(iv) $\mathcal{B}$ is an algebra of subsets of Y.

Moreover, if α is an additive nonnegative extended real valued function on $\mathcal{A}$ and

$$f_\# \alpha = \{(B, \alpha(f^{-1}(B)) : B \in \mathcal{B}\}$$

then

(v) $f_\# \alpha$ is an additive nonnegative extended real valued function on $\mathcal{B}$;

(vi)

$$\mathcal{S}_{f_{\#}\alpha}(\mathcal{B}) = \{t \in \mathbb{R}^Y : t \text{ is constant on } Y \sim \mathbf{rng}\, f \text{ and } t \circ f \in \mathcal{S}_\alpha(\mathcal{A})\};$$

(vii) $I_{f_{\#}\alpha}(t) = I_\alpha(t \circ f)$ for $t \in \mathcal{S}_{f_{\#}\alpha}(\mathcal{B})$;

Proof. Since $2^Y \ni B \mapsto f^{-1}[B] \in 2^X$ preserves set theoretic operations we see that (iv) and (v) hold.

Suppose $t \in \mathbb{R}^Y$. If $t \in \mathcal{S}_{f_{\#}\alpha}(\mathcal{B})$ then $\mathbf{rng}\, t$ is finite so $\mathbf{rng}\, t \circ f$ is finite and, for any $z \in \mathbb{R}$, either $t^{-1}[\{z\}] = Y \sim \mathbf{rng}\, f$ or $t^{-1}[\{z\}] \subset \mathbf{rng}\, f$; in either case, $(t \circ f)^{-1}[\{z\}] = f^{-1}[t^{-1}[\{z\}]] \in \mathcal{A}$ so $t \circ f \in \mathcal{S}_\alpha(\mathcal{A})$. If t is constant on $Y \sim \mathbf{rng}\, f$ and $t \circ f \in \mathcal{S}_\alpha(\mathcal{A})$ then $\mathbf{rng}\, t$ is finite and, for any $z \in \mathbb{R}$, we have $(t \circ f)^{-1}[\{z\}] = f^{-1}[t^{-1}[\{z\}]] = f^{-1}[B]$ for some $B \in \mathcal{B}$. So

$$(t \circ f)^{-1}[\{z\}] \begin{cases} = \emptyset \in \mathcal{A} & \text{if } z \in Y \sim \mathbf{rng}\, f, \\ \in \mathcal{A} & \text{if } z \in \mathbf{rng}\, f. \end{cases}$$

So (vi) holds.

Suppose $t \in \mathcal{S}_{f_{\#}\alpha}(\mathcal{B})$. Then

$$\begin{aligned} I_{f_{\#}\alpha}(t) &= \sum_{z \in \mathbf{rng}\, t} z\, f_{\#}\alpha(t^{-1}[\{z\}]) \\ &= \sum_{z \in \mathbf{rng}\, t} z\, \alpha(f^{-1}[t^{-1}[\{z\}]]) \\ &= \sum_{z \in \mathbf{rng}\, t \circ f} z\, \alpha((t \circ f^{-1})[\{z\}]) \\ &= I_\alpha(t \circ f) \end{aligned}$$

so (vii) holds. □

3. Products.

Suppose X and Y are nonempty sets, $\mathcal{A}$ is an algebra of subsets of X and $\mathcal{B}$ is an algebra of subsets of Y. We let

$$\mathcal{A} \otimes \mathcal{B} = \mathbf{d}(\mathcal{A} \times \mathcal{B}).$$

3.1. Products of algebras.

Proposition 3.1. $\mathcal{A} \otimes \mathcal{B}$ is an algebra of subsets of $X \times Y$.

Proof. Suppose $A, C \in \mathcal{A}$ and $B, D \in \mathcal{B}$. Then

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \in \mathcal{A} \otimes \mathcal{B}$$

so, by ?? and ??, $\mathbf{i}(\mathcal{A}) \subset \mathcal{A}$. Moreover,

$$(A \times B) \sim (C \times D) = \big((A \cap C) \times (B \sim D)\big) \cup \big((A \sim C) \times D\big) \in \mathbf{u}(\mathcal{A} \otimes \mathcal{B})$$

so $\mathbf{c}(\mathcal{A}) \subset \mathbf{d}(\mathcal{A})$. The Proposition now follows from Proposition 1.3(v). □

Remark 3.1. So if $C \in \mathcal{A} \otimes \mathcal{B}$ there is a finite disjointed family $\mathcal{C}$ of $\mathcal{A} \times \mathcal{B}$ such that $C = \cup\mathcal{C}$. In ?? we will provide a $\mathcal{C}$ which is canonical in some sense.

Proposition 3.2. Suppose $\mathcal{R} \subset \mathcal{A}$, $\mathcal{S} \subset \mathcal{B}$, $\mathbf{u}(\mathcal{R}) = \mathcal{A}$ and $\mathbf{u}(\mathcal{S}) = \mathcal{B}$. Then

$$\mathcal{A} \otimes \mathcal{B} = \mathbf{u}(\{R \times S : (R, S) \in \mathcal{R} \times \mathcal{S}\}).$$

Proof. Suppose $C = \{\cup\{A \times B : (A,B) \in \mathcal{F}\}$ where $\mathcal{F}$ is a is a finite subfamily of $\mathcal{A} \times \mathcal{B}$. For each $(A,B) \in \mathcal{F}$ choose a finite subfamily $\mathcal{G}_{A,B}$ of $\mathcal{R}$ such that $\cup\mathcal{G}_{A,B} = A$ and choose a finite subfamily $\mathcal{H}_{A,B}$ of $\mathcal{S}$ such that $\cup\mathcal{H}_{A,B} = B$. If $(A,B) \in \mathcal{F}$ we have

$$A \times B = (\cup\mathcal{G}_{A,B}) \times (\cup\mathcal{H}_{A,B}) = \cup\{G \times H : (G,H) \in \mathcal{G}_{A,B} \times \mathcal{H}_{A,B}\} \in \mathbf{u}(\mathcal{C})$$

where we have set $\mathcal{C} = \{R \times S : (R,S) \in \mathcal{R} \times \mathcal{S}\}$. So $C \in \mathbf{u}(\mathbf{u}(\mathcal{C})) = \mathbf{u}(\mathcal{C})$. □

Corollary 3.1. (??) Suppose $\mathcal{E}$ is a finite family of $\mathcal{A} \otimes \mathcal{B}$-simple functions with values in some vector space Z. Then there are a finite disjointed subfamily $\mathcal{G}$ of $\{A \times B : (A,B) \in \mathcal{A} \times \mathcal{B}\}$ and, for each $s \in \mathcal{E}$, a function $\sigma_s : \mathcal{G} \to Z$ such that

$$s = \sum_{G \in \mathcal{G}} \sigma_s(G) 1_G.$$

Proof. Let

$$\mathcal{F} = \{s^{-1}[\{y\}] : s \in \mathcal{E} \text{ and } y \in Z\}.$$

Using Theorem **??** we let $\mathcal{G}$ be a finite disjointed subfamily $\mathcal{G}$ of $\{A \times B : (A,B) \in \mathcal{A} \times \mathcal{B}\}$ such that

$$F = \cup\{G : G \in \mathcal{G} \text{ and } G \subset F\} \quad \text{whenever } F \in \mathcal{F}.$$

For each $s \in \mathcal{E}$ we let

$$\sigma_s = \{(G,y) : G \in \mathcal{G},\ y \in Z, \text{ and } G \subset s^{-1}[\{y\}]\}.$$

□

3.2. Product integration. Now suppose α and β are additive nonnegative extended real valued functions on $\mathcal{A}$ and $\mathcal{B}$, respectively.

Definition 3.1. For any set W and $u \in W^{X \times Y}$ we define

$$u^{\mathbf{l}} \in (W^Y)^X \quad \text{and} \quad u^{\mathbf{r}} \in (W^X)^Y$$

by setting

$$u^{\mathbf{l}}(x)(y) = u(x,y) = u^{\mathbf{r}}(y)(x) \quad \text{for } (x,y) \in X \times Y.$$

Theorem 3.1. There is a unique additive nonnegative extended real valued function

$$\alpha \otimes \beta$$

on $\mathcal{A} \otimes \mathcal{B}$ such that

$$(\alpha \otimes \beta)(A \times B) = \alpha(A)\beta(B) \quad \text{whenever } (A,B) \in \mathcal{A} \times \mathcal{B}. \tag{10}$$

Moreover, if $u \in \mathcal{S}_{\alpha\otimes\beta}(\mathcal{A} \otimes \mathcal{B})$ then

(i) for each $x \in X$, $u^{\mathbf{l}}(x) \in \mathcal{S}_\beta(\mathcal{B})$ and $I_\beta \circ u^{\mathbf{l}} \in \mathcal{S}_\alpha(\mathcal{A})$;
(ii) for each $y \in Y$, $u^{\mathbf{r}}(y) \in \mathcal{S}_\alpha(\mathcal{A})$ and $I_\alpha \circ u^{\mathbf{r}} \in \mathcal{S}_\beta(\mathcal{B})$;
(iii) $I_\beta(I_\alpha \circ u^{\mathbf{r}}) = I_{\alpha\otimes\beta}(u) = I_\alpha(I_\beta \circ u^{\mathbf{l}})$

Proof. Let $\mathcal{U}$ be the set of $u \in \mathcal{S}_{\alpha\otimes\beta}(\mathcal{A} \otimes \mathcal{B})$ such that (i) and (ii) hold.

Suppose $C \in \mathcal{A} \otimes \mathcal{B}$. By the definition of $\mathcal{A} \otimes \mathcal{B}$ there is a finite disjointed subfamily $\mathcal{F}$ of $\mathcal{A} \times \mathcal{B}$ such that $1_C = \sum_{(A,B)\in\mathcal{F}} 1_{A\times B}$. Then

$$\begin{aligned}
\text{for each } x \in X,\ u^{\mathbf{l}}(x) &= \sum_{(A,B)\in\mathcal{F}} 1_A(x) 1_B \in \mathcal{S}_\beta(\mathcal{B});\\
I_\beta \circ u^{\mathbf{l}} &= \sum_{(A,B)\in\mathcal{F}} \beta(B) 1_A \in \mathcal{S}_\alpha(\mathcal{A})\\
\text{for each } y \in Y,\ u^{\mathbf{r}}(y) &= \sum_{(A,B)\in\mathcal{F}} 1_B(y) 1_A \in \mathcal{S}_\alpha(\mathcal{A});\\
I_\alpha \circ u^{\mathbf{r}} &= \sum_{(A,B)\in\mathcal{F}} \alpha(A) 1_B \in \mathcal{S}_\beta(\mathcal{B})
\end{aligned}$$

so (i) and (ii) hold and

$$I_\beta(I_\alpha^+ \circ u^{\mathbf{r}}). = \sum_{(A,B)\in\mathcal{F}} \alpha(A)\beta(B) = I_\alpha(I_\beta \circ u^{\mathbf{l}}) \tag{11}$$

Thus $1_C \in \mathcal{U}$. Since $\mathcal{U}$ is closed under addition and multiplication by extended nonegative scalars we infer that $\mathcal{U} = \mathcal{S}_{\alpha\otimes\beta}(\mathcal{A} \otimes \mathcal{B})$. It should be clear at this point that there is a unique additive extended real valued function $\alpha \otimes \beta$ on $\mathcal{A} \otimes \mathcal{B}$ such that (10) holds and that, keeping in mind (11), (**??**) holds. □

4. Rectangles and Multirectangles.

4.1. p, q, r**.** If m and n are positive integers we define

$$p_{m,n} : \mathbb{R}^{m+n} \to \mathbb{R}^m, \quad q_{m,n} : \mathbb{R}^{m+n} \to \mathbb{R}^n, \quad r_{m,n} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$$

by requiring that, for each $z \in \mathbb{R}^{m+n}$,

$$(p_{m,n}(z))_i = z_i \quad \text{for } i \in \mathbb{J}(1,m); \qquad (q_{m,J}(z))_j = z_{m+j} \quad \text{for } j \in \mathbb{J}(1,n);$$

and

$$r_{m+n}(p_{m,n}(z), q_{m,n}(z)) = z.$$

Obviously, if $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and $z \in \mathbb{R}^{m+n}$ then

$$r_{m,n}(x,y) = z \ \Leftrightarrow\ x = p_{m,n}(z) \text{ and } y = q_{m,n}(z); \tag{12}$$

in particular, $r_{m,n}$ is univalent with range R^{m+n}.

Definition 4.1. Suppose $n \in \mathbb{N}^+$. We define the family

$$\mathbb{Rect}_n$$

of subsets of $\mathbb{R}^n$ by induction on n as follows. We let $\mathbb{Rect}_1 = \mathbb{Int}$ and we require that

$$\mathbb{Rect}_{n+1} = \{r_{n,1}[Q \times I] : (Q, I) \in \mathbb{Rect}_n \times \mathbb{Rect}_1.$$

We define the family

$$\mathbb{MultiRect}_n$$

of subsets of $\mathbb{R}^n$ by setting

$$\mathbb{MultiRect}_n = \mathbf{d}(\mathbb{Rect}_n).$$

Theorem 4.1. Suppose $n \in \mathbb{N}^+$. Then $\mathbf{i}(\mathbb{Rect}_n) = \mathbb{Rect}_n$, $\mathbf{c}(\mathbb{Rect}_n) \subset \mathbf{d}(\mathbb{Rect}_n)$ and $\mathbb{MultiRect}_n$ is an algebra of subsets of $\mathbb{R}^n$.

Proof. Let $\mathcal{N}$ be the set of $n \in \mathbb{N}^+$ for which the Lemma holds. $1 \in \mathcal{N}$ by ??.

Suppose $n \in \mathcal{N}$, $P, Q \in \mathbb{Rect}_n$ and $I, J \in \mathbb{Rect}_1$. Let $R = r_{n,1}[P \times I]$ and let $S = r_{n,1}[Q \times J]$. Then

$$R \cap S = r_{n,1}[(P \times I) \cap (Q \times J)] = r_{n,1}[(P \cap Q) \times (I \cap J)] \in \mathbb{Rect}_{n+1}$$

so $\mathbf{i}(\mathbb{Rect}_{n+1}) \subset \mathbb{R}^n$ by ??. Moreover,

$$\mathbb{R}^{n+1} \sim R = r_{n,1}[(\mathbb{R}^n \times \mathbb{R}) \sim (P \times I)] = r_{n,1}[\cup \mathcal{S}] = \cup\{r_{n,1}[S] : S \in \mathcal{S}\}$$

where

$$\mathcal{S} = \{(\mathbb{R}^n \sim P) \times I, (\mathbb{R}^n \sim P) \times (\mathbb{R} \sim I)), P \times (\mathbb{R} \sim I)\}$$

so, as $\mathcal{S}$ is disjointed, we infer from ?? that $\mathbf{c}(\mathbb{Rect}_{n+1}) \subset \mathbf{i}(\mathbb{Rect}_{n+1})$. Keeping in mind ?? we see that $n + 1 \in \mathcal{N}$. □

Proposition 4.1. Suppose $R \in \mathbb{Rect}_n$ and R is nonempty. There is a unique n-tuple I of intervals such that

$$R = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n : x_i \in I_i\}.$$

Proof. We leave to the reader to prove the Proposition by a straightforward induction on n. □

Theorem 4.2. There is one and only one additive nonnegative extended real valued function

$$|| \cdot ||_n$$

on $\mathbb{MultiRect}_n$ such that

$$||R||_n = \prod_{i=1}^{n} \mathbf{diam}\, I_i$$

whenever $R \in \mathbb{Rect}_n \sim \{\emptyset\}$ and I is an n-tuple of intervals such that $R = \bigcap_{i=1}^n \{x \in \mathbb{R}^n : x_i \in I_i\}$.

Proof. Let $\mathcal{N}$ be the set of $n \in \mathbb{N}^+$ such that the Theorem holds. $1 \in \mathcal{N}$ by Proposition 1.8 and Theorem 1.2.

Suppose $n \in \mathcal{N}$.

$$\begin{aligned}
&\{r_{n,1}[N] : N \in \mathbb{MultiRect}_n \otimes \mathbb{MultiRect}_1\} \\
&\qquad = \{r_{n,1}[N] : N \in \mathbf{u}(\{R \times S : (R, S) \in \mathbb{Rect}_n \times \mathbb{Rect}_1\})\} \\
&\qquad = \mathbf{u}(\{r_{n,1}[R \times S] : (R, S) \in \mathbb{Rect}_n \times \mathbb{Rect}_1\}) \\
&\qquad = \mathbf{u}(\mathbb{Rect}_{n+1}) \\
&\qquad = \mathbb{MultiRect}_{n+1}.
\end{aligned}$$

Let

$$|| \cdot ||_{n+1} = (r_{n,1})_\#(|| \cdot ||_n \otimes || \cdot ||_1).$$

Suppose I is an $(n+1)$ tuple of nonempty intervals and $R = \bigcap_{i=1}^{n+1} \{x \in \mathbb{R}^{n+1} : x_i \in I_i\}$. Let $Q = \bigcap_{i=1}^n \{x \in \mathbb{R}^n : x_i \in I_i\}$. Then

$$||R||_{n+1} = (r_{n,1})_\#(|| \cdot ||_n \otimes || \cdot ||_1)(R) = ||Q||_n \, ||I_{n+1}||_1 = \prod_{i=1}^{n+1} \mathbf{diam}\, I_i.$$

The uniqueness of $|| \cdot ||_{n+1}$ follows since $\mathbb{MultiRect}_{n+1} = \mathbf{u}(\mathbb{Rect}_{n+1})$. □

Definition 4.2. Let
$$I_n = I_{||\cdot||_n}.$$

Theorem 4.3. $\mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$ is a vector space and I_n is linear. Moreover, if $M \in \mathbb{MultiRect}_n$, $c \in [0,\infty]$ and $s,t \in \mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$ we have

(i) if $M \in \mathbb{MultiRect}_n$ then $I_n(1_M) = ||M||$;
(ii) if $c \in \mathbb{R}^+$ and $s \in \mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$ then $s \wedge c \in \mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$;
(iii) if $s,t \in \mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$ then $s \wedge t \in \mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$;
(iv) if $s,t \in \mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$ and $s \le t$ then $I_n(s) \le I_n(t)$.

Moreover, if $s \in \mathcal{S}_{||\cdot||_n}(\mathbb{MultiRect}_n)$, $m \in \mathbb{N}^+$ and $1 \le m < n$ we have
$$\begin{aligned} I_{n-m}(\mathbb{R}^{n-m} \ni y \mapsto I_m(\mathbb{R}^m \ni x \mapsto s(r_{m,n-m}(x,y)))) \\ &= I_n(s) \\ &= I_m(\mathbb{R}^m \ni x \mapsto I_{n-m}(\mathbb{R}^{n-m} \ni y \mapsto s(r_{m,n-m}(x,y)))). \end{aligned}$$

Proof. Apply ?? and ??. □

Proposition 4.2. The interior, closure and boundary of a multirectangle in $\mathbb{R}^n$ are multirectangles in $\mathbb{R}^n$.

Proof. Suppose $\mathcal{F}$ is a finite family of rectangles in $\mathbb{R}^n$ and $M = \cup\mathcal{F}$. Since $\mathbf{cl}\, M = \cup\{\mathbf{cl}\, R : R \in \mathcal{F}\}$ and since the closure of a rectangle in $\mathbb{R}^n$ is a rectangle in $\mathbb{R}^n$ we find that $\mathbf{cl}\, M$ is a multirectangle in $\mathbb{R}^n$.

Since $\mathbf{bdry}\, M = \mathbf{cl}\, M \cap \mathbf{cl}\,(\mathbb{R}^n \sim M)$ we find that $\mathbf{bdry}\, M$ is a multirectangle in $\mathbb{R}^n$. Finally, since $\mathbf{int}\, M = \mathbf{cl}\, M \sim \mathbf{bdry}\, M$, we find that $\mathbf{bdry}\, M$ is a multirectangle in $\mathbb{R}^n$. □

Proposition 4.3. Suppose M is a multirectangle in $\mathbb{R}^n$. Then
$$||\mathbf{bdry}\, M|| = 0.$$

Proof. Let $\mathcal{F}$ be a finite family of rectangles in $\mathbb{R}^n$ such that $M = \cup\mathcal{F}$. Now any boundary point of M cannot be interior to any $R \in \mathcal{F}$ which implies that $\mathbf{bdry}\, M \subset \cup\{\mathbf{bdry}\, R : R \in \mathcal{F}\}$. Thus
$$||\mathbf{bdry}\, M|| \le \sum_{R\in\mathcal{F}} ||\mathbf{bdry}\, R|| = 0$$
since $||\mathbf{bdry}\, R|| = 0$ for any rectangle in $\mathbb{R}^n$. □

5

Suppose X and Y are nonempty sets.

Definition 5.1. Suppose $C \subset X \times Y$. We let
$$\mathbf{s}_C : X \to 2^Y \quad \text{and} \quad \mathbf{t}_C : Y \to 2^X$$
be such that
$$\mathbf{s}_C(x) = \{y : (x,y) \in C\} \quad \text{whenever } x \in X$$
and
$$\mathbf{t}_C(y) = \{x : (x,y) \in C\} \quad \text{whenever } y \in Y.$$
We let
$$\mathbf{r}_C = \{\mathbf{s}_C^{-1}[\{B\}] \times \mathbf{t}_C^{-1}[\{A\}] : (A,B) \in 2^X \times 2^Y\}.$$

Theorem 5.1. [A Decomposition Theorem.] Suppose $C \subset X \times Y$. Then $\mathbf{r}_C$ is disjointed and

$$C = \bigcup\{D \in \mathbf{r}_C : D \subset C\}.$$

Proof. It is immediate that $\mathbf{r}_C$ is disjointed.

Suppose $(a, b) \in C$. Let $A = \mathbf{t}_C(b)$, let $B = \mathbf{s}_C(a)$ and let $D = \mathbf{s}_C^{-1}[\{B\}] \times \mathbf{t}_C^{-1}[\{A\}]$. Evidently, $(a, b) \in D$ so to complete the proof we need only show that $D \subset C$. Suppose $(c, d) \in D$. Since

$$\{y : (a, y) \in C\} = \mathbf{s}_C(a) = B = \mathbf{s}_C(c) = \{y : (c, y) \in C\}$$

and $(a, b) \in C$ we infer that $(c, b) \in C$. Since

$$\{x : (x, b) \in C\} = \mathbf{t}_C(b) = A = \mathbf{t}_C(d) = \{x : (x, d) \in C\}$$

we infer that $(c, d) \in C$. □

5.1. Construction of $\mathbf{a}(\mathcal{S})$ when $\mathcal{S}$ is finite. *This subsection will not be used elsewhere in the text.*

Definition 5.2. Suppose $\mathcal{S}$ is a nonempty family of subsets of X and $\mathcal{R} \subset \mathcal{S}$. Noting that $\mathcal{R} \cup \mathbf{c}(\mathcal{S} \sim \mathcal{R})$ is nonempty we let

$$\mathbf{b}(\mathcal{R}, \mathcal{S}) = \cap (\mathcal{R} \cap \mathbf{c}(\mathcal{S} \sim \mathcal{R}))\,.$$

Note that $\mathbf{b}(\mathcal{R}, \mathcal{S}) \in \mathbf{a}(\mathcal{S})$ if $\mathcal{S}$ is finite.

We let

$$\mathbf{b}(\mathcal{S}) = \{\mathbf{b}(\mathcal{R}, \mathcal{S}) : \mathcal{R} \subset \mathcal{S}\}.$$

Proposition 5.1. Suppose $\mathcal{S}$ is a nonempty family of subsets of X and $\mathcal{R}$ and $\mathcal{C}$ are subfamilies of $\mathcal{S}$. Then

(i) $\mathbf{b}(\emptyset, \mathcal{S}) = X \sim \cup\mathcal{S}$;
(ii) $\mathbf{b}(\mathcal{S}, \mathcal{S}) = \cap\mathcal{S}$;
(iii) $\mathbf{b}(\mathcal{Q} \cup \mathcal{R}, \mathcal{S}) = \mathbf{b}(\mathcal{Q}, \mathcal{S}) \cup \mathbf{b}(\mathcal{R}, \mathcal{S})$;
(iv) $\mathbf{b}(\mathcal{Q} \cap \mathcal{R}, \mathcal{S}) = \mathbf{b}(\mathcal{Q}, \mathcal{S}) \cap \mathbf{b}(\mathcal{R}, \mathcal{S})$;
(v) $\mathbf{b}(\mathcal{S} \sim \mathcal{R}, \mathcal{S}) = X \sim \mathbf{b}(\mathcal{R}, \mathcal{S})$.

Moreover, $\mathbf{b}(\mathcal{S})$ is a partition of X.

Remark 5.1. Thus

$$2^{\mathcal{S}} \ni \mathcal{R} \mapsto \mathbf{b}(\mathcal{R}, \mathcal{S}) \in 2^X$$

is a morphism of Boolean algebras.

Proof. Exercise for the reader. □

Theorem 5.2. Suppose $\mathcal{S}$ is a finite nonempty family of subsets of X. Then

$$\mathbf{a}(\mathcal{S}) = \mathbf{i}(\mathbf{b}(\mathcal{S})).$$

Proof. □

Remark 5.2. Thus

$$|\mathbf{a}(\mathcal{S})| \le 2^{2^{|\mathcal{S}|}}.$$

Moreover,

$$S = \cup\{R \in \mathbf{b}(\mathcal{S}) : R \subset S\} \text{whenever } S \in \mathcal{S}.$$

Exercise 5.1. Draw the typical Venn diagram with three sets A, B, C and picture the members of $\mathbf{a}(\{A, B, C\})$.

Example 5.1. Suppose n is a positive integer. Let $X = \{0,1\}^n$ and let

$$\mathcal{S} = \{S_i : i \in \{1,\dots,n\}\}$$

where for each $i \in \{1,\dots,n\}$ we have set

$$S_i = \{a \in X : x_i = 1\}.$$

Then

$$|\{\mathbf{b}(\mathcal{R},\mathcal{S}) : \mathcal{R} \subset \mathcal{S}\}| = 2^n$$

and

$$|\mathbf{a}(\mathcal{S})| = 2^{2^n}$$

which implies that

$$\mathbf{a}(\mathcal{S}) = 2^X.$$

In this case, the aforementioned morphism of Boolean algebras is an isomorphism.

The preceding Theorem together with The following Proposition provides what could reasonable called a construction of the algebra of sets generated by a family of subsets of X.

Theorem 5.3. Suppose $C \subset X \times Y$. The following are equivalent:

(i) $C \in \mathcal{A} \otimes \mathcal{B}$;
(ii) $\mathbf{s}_C \in \mathcal{S}(\mathcal{A}, 2^Y)$ and $\mathbf{t}_C \in \mathcal{S}(\mathcal{B}, 2^X)$
(iii) $\mathbf{r}_C = \{A \times B : (A,B) \in \mathcal{F}\}$ for some finite subfamily $\mathcal{F}$ of $\{\mathcal{A} \times \mathcal{B}$;

Moreover, if C and $\mathcal{F}$ are as in (i) and (iii) then

$$1_C = \sum_{(A,B)\in\mathcal{F}} 1_{A\times B}.$$

Proof. Suppose $\mathcal{F}$ is a finite subfamily of $\mathcal{A} \times \mathcal{B}$ and $C = \cup\mathcal{F}$. For any $x \in X$ we have

$$\mathbf{s}_C(x) = \cup\{\mathbf{s}_{A\times B}(x) : (A,B) \in \mathcal{F}\} = \cup\{B : (A,B) \in \mathcal{F} \text{ and } x \in A\}$$

and for any $y \in Y$ we have

$$\mathbf{t}_C(y) = \cup\{\mathbf{t}_{A\times B}(y) : (A,B) \in \mathcal{F}\} = \cup\{A : (A,B) \in \mathcal{F} \text{ and } y \in B\}.$$

Thus (i) implies (ii).

It is immediate that (ii) implies (iii).

That (iii) implies (i) follows immediately from the Theorem 5.1. □

5.2. Construction of $\mathbf{a}_\sigma(\mathcal{S})$. Suppose $\mathcal{S} \subset 2^X$.

Let ω_1 be the first uncountable ordinal and let

$$\Omega = \{x : x \text{ is an ordinal and } x \le \omega_1\}.$$

We construct a function $f : \Omega \to 2^X$ by induction as follows. Let

$$\mathcal{G} = \{g : \text{for some } x,\ x \in \Omega \text{ and } g : \mathbb{I}(x) \to 2^X\}$$

and let

$$G : \mathcal{G} \to 2^X$$

be such that if $x \in \Omega$ and $g : \mathbb{I}(x) \to 2^X$ then

$$G(g) = \begin{cases} \mathcal{S} \cup \{0\} & \text{if } x = 0, \\ \cup_{w\in\mathbb{I}(x)} g(w) & \text{if } x \ne 0 \text{ and } x = \mathbf{S}(w) \text{ for no } w \in \mathbb{I}(x), \\ g(w) \cup \mathbf{c}(g(w)) \cup \mathbf{u}_\sigma(g(w)) & \text{if } x = \mathbf{S}(w) \text{ for some } w \in \mathbb{I}(x). \end{cases}$$

Let $f : \Omega \to 2^X$ be such that

$$f(x) = G(f|\mathbb{I}(x)) \quad \text{for } x \in \Omega.$$

So

$$f(x) = \begin{cases} \mathcal{S} \cup \{0\} & \text{if } x = 0, \\ \cup_{w \in \mathbb{I}(x)} f(w) & \text{if } x \neq 0 \text{ and } x = \mathbf{S}(w) \text{ for no } w \in \mathbb{I}(x), \\ f(w) \cup \mathbf{c}(f(w)) \cup \mathbf{u}_\sigma(f(w)) & \text{if } x = \mathbf{S}(w) \text{ for some } w \in \mathbb{I}(x). \end{cases}$$

Theorem 5.4. $f(\omega_1) = \mathbf{a}_\sigma(\mathcal{S})$.

Proof. Since ω_1 is a limit ordinal we have

$$f(\omega_1) = \cup_{w \in \mathbb{I}(\omega_1)} f(w).$$

Suppose $A \in f(\omega_1)$. Then $A \in f(w)$ for some $w \in \mathbb{I}(\omega_1)$ so

$$X \sim A \in \mathbf{c}(f(w)) \in f(\mathbf{S}(w)) \subset f(\omega_1).$$

Suppose $\mathcal{A} \subset f(\omega_1)$ and $\mathcal{A}$ is a countable. For each $A \in \mathcal{A}$ let $\alpha(A)$ be the first $w \in \mathbb{I}(\omega_1)$ such that $A \in f(w)$. Then

$$\cup \mathcal{A} \subset \cup_{A \in \mathcal{A}} f(\alpha(A)) \subset f(\omega_1).$$

So $f(\omega_1)$ is a σ-algebra of subsets of X.

Suppose $\mathcal{A}$ is a σ-algebra of subsets of X and $\mathcal{S} \subset \mathcal{A}$. Let

$$\Gamma = \{\alpha \in \mathbb{I}(\omega_1) : f(\alpha) \subset \mathcal{A}\}.$$

Then $f(0) = \mathcal{S} \subset \mathcal{A}$. Suppose β is an ordinal, $0 < \beta \leq \omega_1$ and $f(\alpha) \subset \mathcal{A}$ if $0 \leq \alpha < \beta$. Since $\{\alpha : \alpha < \beta\}$ is countable and $\mathbf{u}_\sigma(\mathcal{A}) \subset \mathcal{A}$ we find that $f(\beta) = \cup_{\alpha \in \mathbb{I}(\beta)} f(\alpha) \subset \mathcal{A}$. So $\Gamma = \Omega$ and $f(\omega_1) \subset \mathcal{A}$. □