1. Trees; Context Free Grammars.

1.1. Trees.

**Definition 1.1.** By a tree we mean an ordered triple
\[ T = (N, o, p) \]
such that
(i) \( N \) is a finite set;
(ii) \( o \in N \);
(iii) \( p : N \sim \{o\} \rightarrow N \);
(iv) if \( \nu \in N \sim \{o\} \) and \( n \in \mathbb{N}^+ \) then \( (\nu, \nu) \notin p^n \).

One can relax the condition that \( N \) be finite but we will have no use for infinite trees.

The members of \( N \) are called nodes. \( o \) is called the root node. If \( \nu \in N \sim \{o\} \) we call \( p(\nu) \) the parent of \( \nu \). If \( \nu \in N \) we let \( c(\nu) = p^{-1}[\nu] = \{\mu \in N : p(\mu) = \nu\} \) and call the members of this set the children of \( \nu \). A node which has children is called an interior node. A node which has no children is called a leaf node. We let
\[ i(T) = \{\nu \in N : \nu \text{ is an interior node of } T\} \]
and we let
\[ l(T) = \{\nu \in N : \nu \text{ is a leaf node of } T\}. \]

For each \( \mu \in N \) we let
\[ ([\mu]) = \{(n, \nu) \in \mathbb{N}^+ \times N : (\mu, \nu) \in p^n\} \]
and we let
\[ d(\mu) = \text{card} ([\mu]); \]

**Proposition 1.1.** Suppose \( \mu \in N \). Then
(i) \( ([\mu]) \) is a univalent function;
(ii) \( d(\mu) = 0 \) if and only if \( \mu = o \);
(iii) \( 0 < d(\mu) \leq \text{card} N \) if \( \mu \neq o \);
(iv) if \( \mu \neq o \) then \( \text{dmm} ([\mu]) = I[1, d(\mu)] \).

**Proof.** That \( ([\mu]) \) is a function follows directly from the fact that \( p^n \) is a function for any \( n \in \mathbb{N}^+ \). Suppose \( (n, \nu) \in ([\mu]) \) for \( i = 1, 2 \) and \( n_2 \geq n_1 \). Then \( p^{n_2-n_1}(p^{n_1}(\mu)) = p^{n_2}(\mu) = p^{n_1}(\mu) \) so, by (iv) of Definition 1.1, \( n_2 = n_1 \). So \( ([\mu]) \) is univalent and (i) holds.

If \( \mu = o \) then \( ([\mu]) = \emptyset \) so (ii) holds. (iii) follows from (i) and (ii).

Suppose \( \mu \neq o \). Then \( \mu \in \text{dmm} p \) so \( (1, p(\mu)) \in ([\mu]) \) so \( 1 \in \text{dmm} ([\mu]) \). Suppose \( n \in \text{dmm} ([\mu]) \) and \( 1 < n \). Let \( \nu \) be such that \( (n, \nu) \in ([\mu]) \). Then \( \nu = p^n(\mu) \) which implies that \( (\mu, p^{n-1}(\mu)) \in p^{n-1} \) so \( (n - 1, p^{n-1}(\mu)) \in ([\mu]) \) so \( n - 1 \in \text{dmm} ([\mu]) \).

Since \( ([\mu]) \) is univalent, (iv) follows. \( \Box \)

Suppose \( \mu \in N \sim \{o\} \). We let
\[ A(\mu) = \text{rng} ([\mu]) \]
and we call the members of this set ancestors of \( \mu \). If \( \nu \in A(\mu) \) we say that \( \mu \) is a descendant of \( \nu \).
1.2. Suppose \((\mathcal{N}, o, p)\) is a tree. For each \(d \in \mathbb{N}\) we let
\[
\mathcal{N}^{(d)} = p^{-d}\{o\}.
\]

**Proposition 1.2.** Suppose \(d_i \in \mathbb{N}\), \(i = 1, 2\), and \(d_1 \neq d_2\). Then
\[
\mathcal{N}^{(d_1)} \cap \mathcal{N}^{(d_2)} = \emptyset.
\]

**Proof.** Suppose \(\mu \in \mathcal{N}^{(d_1)} \cap \mathcal{N}^{(d_2)}\). Then
\[
p^{d_2 - d_1}(p^{d_1}(\mu)) = p^{d_2}(\mu) = p^{d_1}(\mu)
\]
which is incompatible with (iv). \(\Box\)

1.3. Subtrees.

**Definition 1.2.** We say the tree \(U = (\mathcal{O}, \sigma, q)\) is a subtree of \(T = (\mathcal{N}, o, p)\) if \(\sigma \in \mathcal{O} \subset \mathcal{N}\) and \(q = p|\mathcal{O} \sim \{\sigma\}\).

Given \(\nu \in \mathcal{N}\) let \(\mathcal{N}_\nu\) be the set whose members are \(\nu\) and the descendants of \(\nu\), let \(p_\nu = p|\mathcal{N}_\nu \sim \{\nu\}\) and note that
\[
\mathcal{T}_\nu = (\mathcal{N}_\nu, \nu, p_\nu)
\]
is a subtree of \(\mathcal{T}\) which we call the subtree associated to the node \(\nu\).

1.4. Isomorphisms. Suppose \(\mathcal{T}_i = (\mathcal{N}_i, o_i, p_i), i = 1, 2\) are trees and \(\iota : \mathcal{N}_1 \rightarrow \mathcal{N}_2\). We say \(\iota\) is an isomorphism from \(\mathcal{T}_1\) to \(\mathcal{T}_2\) if \(\iota\) is univalent, \(\text{rng } \iota = \mathcal{N}_2\) and
\[
p_2 \circ \iota = \iota \circ p_1;
\]
it follows that \(\iota(o_1) = o_2\) and that \(\iota^{-1}\) is an isomorphism from \(\mathcal{T}_2\) to \(\mathcal{T}_1\).

1.5. Orderings of trees. Suppose \(\mathcal{T} = (\mathcal{N}, o, p)\) is a tree. We say \(\prec\) is an ordering of \(\mathcal{T}\) if
\begin{enumerate}
\item \(\prec\) is well ordering of \(\mathcal{N}\);
\item \(o \prec \nu\) for all \(\nu \in \mathcal{N} \sim \{o\}\);
\item if \(\mu, \nu \in \mathcal{N} \sim \{\nu\}\) and \(\mu \prec \nu\) then either \(p(\mu) = p(\nu)\) or \(p(\mu) \prec p(\nu)\).
\end{enumerate}

Suppose \(\prec\) is an ordering of \(\mathcal{T}\).

**Proposition 1.3.** Suppose \(\mu, \nu \in \mathcal{N} \sim \{\nu\}\) and \(p(\mu) \prec p(\nu)\). Then \(\mu \prec \nu\).

**Proof.** Since \(p(\mu) \prec p(\nu)\) we have \(p(\mu) \neq p(\nu)\) and this implies \(\mu \neq \nu\). So either (I) \(\mu \prec \nu\) or (II) \(\nu \prec \mu\). If (III) held we would infer from (iii) that \(p(\nu) \prec p(\mu)\) and this is impossible. \(\Box\)

**Proposition 1.4.** Suppose \(\mu, \nu \in \mathcal{N} \sim \{\nu\}\) and \(d(\mu) < d(\nu)\). Then \(\mu \prec \nu\).

**Proof.** Let \(\mathcal{M}\) be the set of \(m \in \mathcal{M}\) such that the Proposition holds when \(d(\mu) = m\). Then \(0 \in \mathcal{M}\) by (ii). Suppose \(m \in \mathcal{M}\) and \(d(\mu) = m + 1\). Then \(d(p(\mu)) = d(\mu) - 1 < d(\nu) - 1 = d(p(\nu))\) and \(d(p(\mu)) = (m + 1) - 1 = m \in \mathcal{M}\) so \(p(\mu) \prec p(\nu)\). From the preceding Proposition we find that \(\mu \prec \nu\). \(\Box\)

**Corollary 1.1.** Suppose \(\mu \in \mathcal{N} \sim \{\nu\}\). Then \(p(\mu) \prec \nu\).

**Theorem 1.1.** Suppose for each \(\xi \in \mathcal{N}\) we are given a well ordering \(\prec_\xi\) of \(c(\xi)\). Then there is a ordering \(\prec\) of \(\mathcal{T}\) such that
\[
\mu \prec \nu \iff \mu \prec_\xi \nu\quad \text{for } \xi \in \mathcal{N}\text{ and } \mu, \nu \in c(\xi).
\]
Suppose for each $2 \in \mathcal{N}$ and for each $d \in I[0, D]$ let $\mathcal{N}_d = \{ \mu \in \mathcal{N} : d(\mu) \leq d\}$ and let $\mathcal{N}_{[d]} = \bigcup_{e=d}^{D} \mathcal{N}_e$. For each $d \in I[0, D]$ we define the relation $W_d$ on $\mathcal{N}_{[d]}$ by induction on $d$ as follows. We let $W_0 = \emptyset$. If $d \in I[0, D)$ we let

$$W_{d+1} = W_d \cup W_d^1 \cup W_d^2 \cup W_d^3$$

where

$$W_d^1 = \mathcal{N}_{[d]} \times \mathcal{N}_{d+1};$$
$$W_d^2 = \{ (\mu, \nu) \in \mathcal{N}_{d+1} \times \mathcal{N}_{d+1} : (p(\mu), p(\nu)) \in W_d \};$$
$$W_d^3 = \{ (\mu, \nu) \in \mathcal{N}_{d+1} \times \mathcal{N}_{d+1} : \text{for some } \xi, \xi \in \mathcal{N}, p(\mu) = \xi = p(\nu) \text{ and } \mu \prec \xi \nu \}.$$ 

One verifies by induction on $d \in I[0, D]$ that $W_d$ is a well ordering of $\mathcal{N}_{[d]}$. □

**Theorem 1.2.** Suppose for each $\mu \in \mathcal{N}$ we are given a well ordering $\prec_\mu$ of $c(\mu)$. Then there is one and only one ordering $\prec$ of $\mathcal{T}$ such that, for each $\xi \in \mathcal{N}$ and each $\mu, \nu \in c(\xi),$

$$(1) \quad \mu \prec_\xi \nu \iff \mu \prec \nu.$$ 

**Proof.** We define a relation $\prec$ on $\mathcal{N}$ as follows. Suppose $\mu, \nu \in \mathcal{N}$ and $\mu \neq \nu$. If $\mu \in A(\nu)$ then $\mu \prec \nu$. If $\nu \in A(\mu)$ then $\nu \prec \mu$.

Suppose $\mu \notin A(\nu)$ and $\nu \notin A(\mu)$. Let $\xi$ be the deepest common ancestor of $\mu$ and $\nu$. Let $m, n \in \mathbb{N}^+$ be such that $p^{m}(\mu) = \xi = p^{n}(\nu)$. Then $\{p^{m-1}(\mu), p^{n-1}(\nu)\} \subset c(\xi)$. Then $\mu \prec \nu$ if $p^{m-1} \prec_\xi p^{n-1}(\nu)$ and $\nu \prec \mu$ if $p^{n-1}(\nu) \prec_\xi p^{m-1}(\xi)$.

We leave to the reader the straightforward proof that $\prec$ is an ordering of $\mathcal{T}$ as well as the straightforward verification using ?? and ?? that it is the only ordering of $\mathcal{T}$ satisfying (1). □

Suppose $\mathcal{O}_i = (\mathcal{N}_i, o_i, p_i)$, $i = 1, 2$ are ordered trees and $\iota : \mathcal{N}_1 \to \mathcal{N}_2$. We say $\iota$ is an isomorphism from $\mathcal{O}_1$ to $\mathcal{O}_2$ if $\iota$ is an isomorphism from $(\mathcal{N}_1, o_1, p_1)$ to $(\mathcal{N}_2, o_2, p_2)$ and

$$\mu, \nu \in \mathcal{N}_1 \text{ and } \mu \prec_1 \nu \Rightarrow \iota(\mu) \prec_2 \iota(\nu).$$

Suppose $\mathcal{O} = (\mathcal{N}, o, p, \prec)$ is an ordered tree and $\mathcal{U} = (\mathcal{O}, \sigma, q)$ is a subtree of $\mathcal{T} = (\mathcal{N}, o, p)$. Let

$$\prec = \{(\mu, \nu) \in \mathcal{O} \times \mathcal{O} : \mu \prec \nu\}$$

and note that $(\mathcal{O}, \sigma, q, \prec)$ is an ordered tree which we call the ordered tree associated to $\mathcal{U}$. In particular, if $\nu \in \mathcal{N}$, $\mathcal{O} = \mathcal{N}_\nu$, $\sigma = \nu$ and $q = q_\nu$ we call this ordered tree the ordered tree associated to $\nu$.

**1.6. Tree codes.** Suppose $N \in \mathbb{N}^+$. We let

$$\mathcal{T}(N)$$

be the set of functions $P : I[1, N] \to I[0, N]$ such that

$$(2) \quad P^{-1}([0]) \neq \emptyset;$$

$$(3) \quad i \in I[1, N] \Rightarrow P(i) < i;$$

$$(4) \quad i, j \in I[1, N] \text{ and } i < j \Rightarrow P(i) \leq P(j).$$

**Proposition 1.5.** Suppose $N \in \mathbb{N}^+$ and $P \in \mathcal{T}(N)$. Then $\mathcal{T} = (I[1, N], P, 0)$ is a tree and the standard ordering of $I[1, N]$ is a ordering of $\mathcal{T}$. 

Suppose \( T = (N, o, p) \) is a tree and \( \prec \) is tree order for \( T \). Let \( N = \text{card} N \) and let

\[
W : I[0, N) \rightarrow N
\]

be such that

\[
i, j \in I[0, N) \text{ and } i < j \Rightarrow W(i) \prec W(j).
\]

Note that \( W(0) = o \) and \( W \) is univalent; in particular, \( W^{-1} \) is a function. Let

\[
P : I[1, N) \rightarrow N
\]

be such that

\[
P(i) = W^{-1}(p(W(i)) \text{ for } i \in I[1, N).
\]

We call \( P \) the **tree code** for \( (T, \prec) \).

**Proposition 1.6.** \( P \in T(N) \).

**Proof.** Suppose \( i \in I(1, N) \). Let \( \mu = W(i) \). Then \( W(P(i)) = p(W(i)) = p(\mu) \prec \mu = W(i) \) so \( P(i) < i \).

Suppose \( i, j \in I(1, N) \), \( i < j \) and \( P(i) \neq P(j) \). Let \( \mu = W(i) \) and let \( \nu = W(j) \). Then \( \mu \prec \nu \). We have \( W(P(i)) = p(W(i)) = p(\mu) \) and \( W(P(j)) = p(W(j)) = p(\nu) \).

Since \( P(i) \neq P(j) \) we have \( \mu \neq \nu \) so \( p(\mu) < p(\nu) \) so \( P(i) < P(j) \).

\( \square \)

**Theorem 1.3.** Suppose, for \( i = 1, 2 \), \( T_i = (N_i, p_i, o_i) \) is a tree and \( \prec_i \) is an ordering of \( T_i \). Then \( T_i, i = 1, 2, \) are \((\prec_1, \prec_2)\) isomorphic if and only if they have the same tree codes.

**Proof.** Let \( \iota : N_1 \rightarrow N_2 \) be a \((\prec_1, \prec_2)\) isomorphism and let \( T_i, i = 1, 2, \) be the \( \prec_i \) tree codes of \( T_i \) respectively. Then

\[
P_2 = W_2^{-1} \circ p_2 \circ W_2
\]

\[
= W_2^{-1} \circ l \circ p_1 \circ l^{-1} \circ W_2
\]

\[
= W_2^{-1} \circ l \circ W_1 \circ p_1 \circ W_1^{-1} \circ l^{-1} \circ W_2
\]

\[
= (W_2^{-1} \circ l \circ W_1) \circ p_1 \circ (W_1^{-1} \circ l^{-1} \circ W_2);
\]

Since \( W_2^{-1} \circ l \circ W_1 \) and \( W_1^{-1} \circ l^{-1} \circ W_2 \) are order preserving maps of \( I[1, N) \) to itself they are equal to the identity map of \( I[1, N) \).

\( \square \)

### 1.7. Growing trees

Suppose

\[
T = (N, o, p)
\]

is a tree and

\[
\mathcal{O} \text{ and } \mathcal{P}
\]

satisfy the following conditions:

(i) \( \mathcal{O} \) and \( \mathcal{P} \) are functions with domain the leaf nodes of \( T \);

(ii) for each \( \nu \in N \), \( \mathcal{O}(\nu), \nu, \mathcal{P}(\nu) \) is a tree;

(iii) the family

\[
\{i(T)\} \cup \{\mathcal{O}(\nu) : \nu \in \mathcal{I}(T)\}
\]

is disjointed.
Let 
\[ U = \mathcal{I}(T) \cup \left( \bigcup \{ \mathcal{O}(\nu) : \nu \in \mathcal{I}(T) \} \right) \]
and let 
\[ q = \mathcal{P} \cup \left( \bigcup \{ \mathcal{P}(\nu) : \nu \in \mathcal{I}(T) \} \right). \]

We leave it as an exercise for the reader to verify that 
\[ (U, o, q) \]
is a tree; that \( N \subseteq U \); and that, for each \( \nu \in \mathcal{I}(T) \), \( (\mathcal{O}(\nu), \nu, \mathcal{P}(\nu)) \) is the subtree associated to the node \( \nu \) of \( U \).

Now let us suppose that 
\[ (N, o, p, <) \]
is an ordered tree and that, for each \( \nu \in \mathcal{I}(T) \), \( <_\nu \) is such that 
\[ (\mathcal{O}(\nu), \nu, \mathcal{P}(\nu), <_\nu) \]
is an ordered tree. We leave it to the reader to verify that there is one and only one
\[ \prec \]
such that
(i) \( (U, o, q, <) \) is an ordered tree;
(ii) if \( \mu, \xi \in N \) and \( \mu < \xi \) then \( \mu <_\nu \xi \);
(iii) if \( \nu \in \mathcal{I}(T) \), \( \mu, \xi \in \mathcal{O}(\nu) \) and \( \mu <_\nu \xi \) then \( \mu < \xi \).

### 2. Context Free Grammars.

**Definition 2.1.** By a context free grammar ordered triple
\[ G = (T, N, s, \mathcal{P}) \]
such that
(i) \( T \) is a set;
(ii) \( N \) is a set, \( T \cap N = \emptyset \) and \( s \in N \).
(iii) \( \mathcal{P} \subseteq N \times (T \cup N)^* \).

The members of \( T \) are called tokens or terminal symbols. The members of \( N \) are called nonterminals or nonterminal symbols. \( s \) is called the start symbol. The members of \( \mathcal{P} \) are called productions. Instead of writing \( (r, s) \in \mathcal{P} \) one often writes
\[ r := \epsilon \quad \text{if} \ |s| = 0 \]
and
\[ r := s_0 \ s_1 \ \cdots \ s_{|s|-1} \quad \text{if} \ |s| > 0. \]
If \( r \in N, \ n \in \mathbb{N}^+ \) and \( s_0, \ldots, s_{n-1} \in (T \cup N)^* \) one often writes
\[ r := s_0 \mid s_1 \mid \cdots \mid s_{n-1} \]
instead of
\[(r, s_i) \in \mathcal{P}, \ i \in I(n).\]

Obviously, if \( (T, N, s, \mathcal{P}) \) is a context free grammar then so is \( (T, N, t, \mathcal{P}) \) if \( t \in N \).
Definition 2.2. A parse tree $Q$ for the context free grammar $G = (T, N, s, \mathcal{P})$ is an ordered quintuple
\[
(N, o, p, <, f)
\]
such that
\begin{enumerate}[(i)]
\item $(N, o, p, <)$ is an ordered tree;
\item $f : N \rightarrow T \cup N$;
\begin{enumerate}[(a)]
\item if $\nu \in \mathbf{i}(T)$ then $f(\nu) \in N$;
\item if $\nu \in \mathbf{l}(T)$ then $f(\nu) \in T$;
\end{enumerate}
\item if $\nu \in \mathbf{i}(T)$ and $\nu$ has $m$ children
\[
\mu_0 < \mu_2 < \ldots < \mu_{m-1}
\]
then
\[
(f(\nu), (f(\mu_0), \ldots, f(\mu_{m-1}))) \in \mathcal{P}.
\]
Notice that the notion of parse tree is independent of $s$.
We define
\[
< Q > \in (T)^*
\]
as follows. Let
\[
\mathcal{L} = \{\nu \in \mathbf{l}(T) : f(\nu) \neq \epsilon\}
\]
and let
\[
n = |\mathcal{L}|.
\]
If $n = 0$ we let
\[
< Q > = \epsilon
\]
and if $n > 0$ and
\[
\nu_0 < \nu_1 < \cdots < \nu_{n-1}
\]
are the members of $\mathcal{L}$ we let
\[
< Q > = (f(\nu_0))|f(\nu_1)| \cdots |f(\nu_{n-1})|.
\]
For each $t \in N$ we let
\[
\mathbf{L}(G, t)
\]
be the set of $< Q >$ as above where $f(o) = t$. We let
\[
\mathbf{L}(G) = \mathbf{L}(G, s)
\]
and we call this language on the alphabet $T$ the language generated by $G$.

Definition 2.3. We say the context free grammar $G$ is **good** if $Q_i = (N_i, o_i, p_i, <_i, f_i)$, $i = 1, 2$, are parse trees such that $< Q_1 > = < Q_2 >$ then the ordered trees $(N_i, o_i, p_i, <_i)$ are isomorphic; recall that this is the case if and only if they have the same tree codes. We say $G$ is **bad** if it is not good.
2.1. A bad grammar. Let $B$ be the context free grammar defined as follows. Let

$$T = \mathbb{N} \cup \{-, +, \ast\}$$

and let

$$N = \{\text{expr}\}.$$  

Let \text{expr} be the start symbol. Let the productions be given by

\begin{align*}
\text{expr} & := j \quad \text{for } j \in \mathbb{N} \\
\text{expr} & := - \text{ expr} \\
\text{expr} & := \text{expr} + \text{expr} \\
\text{expr} & := \text{expr} \ast \text{expr}
\end{align*}

Note that there are an infinite number of productions.

This grammar derives the string $27 + 5 \ast 298$ with two nonisomorphic parse trees so it is bad.

There are a number of ways to deal with this problem. One is to introduce parentheses which we now do. As we shall see, there are other ways to deal with this problem.