1. Trees; context free grammars.

1.1. Trees.

Definition 1.1. By a rooted tree we mean an ordered triple
\[ T = (\mathcal{N}, o, p) \]
such that
(i) \( \mathcal{N} \) is a set;
(ii) \( o \in \mathcal{N} \);
(iii) \( p : \mathcal{N} \sim \{o\} \to \mathcal{N} \);
(iv) if \( \nu \in \mathcal{N} \sim \{o\} \) there is \( d \in \mathbb{N}^+ \) such that \( (\nu, o) \in p^d \).

One calls \( o \) the root of \( T \).

The members of \( \mathcal{N} \) are called nodes. \( o \) is called the root node. If \( \nu \in \mathcal{N} \sim \{o\} \) we call \( p(\nu) \) the parent of \( \nu \). If \( \nu \in \mathcal{N} \) we let
\[ c(\nu) = p^{-1}[\{\nu\}] = \{ \mu \in \mathcal{N} \sim \{o\} : p(\mu) = \nu \} \]
and call the members of this set the children of \( \nu \). A node which has children is called an interior node. A node which has no children is called a leaf node. We let
\[ i \quad \text{and} \quad 1 \]
be the set of interior nodes of \( T \) and the set of leaf nodes of \( T \), respectively.

Definition 1.2. Let
\[ d = \bigcup_{d \in \mathbb{N}} p^{-d}[\{o\}] \times \{d\} \]

Proposition 1.1. \( d \) is a function with domain \( \mathcal{N} \).

Proof. \((o, 0) \in p^{-0}[\{o\}] \times \{0\} \) so \( o \in \text{dom } d \). If \( (\nu, o) \in p^d \) then \( (\nu, o) \in p^{-d}[\{o\}] \times \{d\} \). This is \( \text{dom } d \).

Suppose \( \nu \in \mathcal{N} \) and, for \( i = 1, 2 \), \( d_i \in \mathbb{N} \) and \( (\nu, d_i) \in d \). Then \( \nu \in p^{-d_i}[\{o\}] \) for \( i = 1, 2 \). Suppose \( d_2 > d_1 \). Then
\[ (o, p^{d_2-1}(\nu)) = (o, p^{d_2-d_1}(p^{d_1}(\nu))) = (o, p^{d_2-d_1}(o)) \]
so \( o \in \text{dom } p^{d_2-d_1} \subset \text{dom } p \) which is incompatible with \( \text{dom } p = \mathcal{N} \sim \{o\} \).

Corollary 1.1. \( \{p^{-d}[\{o\}] : d \in \mathbb{N}\} \) is a partition of \( \mathcal{N} \). Moreover,
\[ p^{-n}[\{o\}] = d^{-1}[\{n\}] \quad \text{for } n \in \mathbb{N} \]

Suppose \( \mu \in \mathcal{N} \sim \{o\} \). We let
\[ A(\mu) = \text{rng } (\mu) \]
and we call the members of this set ancestors of \( \mu \). If \( \nu \in A(\mu) \) we say that \( \mu \) is a descendant of \( \nu \).

For each \( \mu \in \mathcal{N} \) we let
\[ A(\mu) = \bigcup_{n \in \mathbb{N}} \{ \nu : (\mu, \nu) \in p^n \} \sim \{\mu\} \]
and we let
\[ D(\mu) = \bigcup_{n \in \mathbb{N}} \{ \nu : \nu \in p^{-n}[\{\mu\}] \} \]
The members of \( A(\mu) \) are called ancestors of \( \mu \) and the members of \( D(\mu) \) are called descendants of \( \mu \).
Suppose for each \( \prec \).

We say the rooted tree \( U = (O, \sigma, q) \) is a rooted subtree of \( T = (N, o, p) \) if \( O \subseteq N \) and \( q = p|O \sim \{\sigma\} \).

Given \( \nu \in N \) one easily verifies that 

\[
(\{\nu\} \cup D(\nu), \nu, p|D(\nu))
\]

is a rooted subtree of \( T \) which we call the rooted subtree associated to the node \( \nu \).

1.3. Isomorphisms. Suppose \( T_i = (N_i, o_i, p_i), i = 1, 2 \) are rooted trees and \( \iota : N_1 \rightarrow N_2 \). We say \( \iota \) is an isomorphism from \( T_1 \) to \( T_2 \) if \( \iota \) is univalent, \( \text{rng}\, \iota = N_2 \) and

\[
p_2 \circ \iota = \iota \circ p_1;
\]

it follows that \( \iota(o_1) = o_2 \) and that \( \iota^{-1} \) is an isomorphism from \( T_2 \) to \( T_1 \).

1.4. Tree orderings. Suppose \( T = (N, o, p) \) is a tree. We say \( \prec \) is a tree ordering of \( T \) if

(i) \( \prec \) is well ordering of \( N \);
(ii) \( (o, N \sim \{o\}) \subset \prec \) for all \( \nu \in N \sim \{o\} \);
(iii) if \( \mu, \nu \in N \) and \( \mu \prec \nu \) then \( D(\mu) \times D(\nu) \subset \prec \).

Lemma 1.1. Suppose \( \mu, \nu \in N \) and \( \mu \neq \nu \). There is a one and only one ordered triple \( (\lambda, m, n) \in N \times N \in N \) such that \( (\mu, \lambda) \in p^m \), \( (\nu, \lambda) \in p^n \) and such that if \( (k, l) \in N \times N \), \( k < m \) and \( l < n \) then \( p^k(\mu) \neq p^l(\nu) \).

Theorem 1.1. Suppose for each \( \xi \in N \) we are given a well ordering \( \prec_{\xi} \) of \( c(\xi) \). Then there is one and only one tree ordering \( \prec \) of \( T \) such that, for each \( \xi \in N \) and each \( \mu, \nu \in c(\xi) \),

\[
\mu \prec_{\xi} \nu \iff \mu \prec \nu.
\]

Proof. Suppose \( \mu, \nu \in N \) and \( \mu \neq \nu \). By the preceding Lemma there is a one and only one ordered triple \( (\lambda, m, n) \in N \times N \in N \) such that \( (\mu, \lambda) \in p^m \), \( (\nu, \lambda) \in p^n \) and such that if \( (k, l) \in N \times N \), \( k < m \) and \( l < n \) then \( p^k(\mu) \neq p^l(\nu) \). If \( k = 0 \) then \( \nu \in D(\mu) \) and we place \( (\mu, \nu) \) in \( \prec \). If \( l = 0 \) then \( \mu \in D(\nu) \) and we place \( (\nu, \mu) \) in \( \prec \). If \( k \neq 0 \) and \( l \neq 0 \) then \( p^{-1}(\mu) \) and \( p^{-1}(\nu) \) are children of \( \lambda \) and we place \( (\mu, \nu) \in \prec \) if \( p^{-1}(\mu) \prec_{\lambda} p^{-1}(\nu) \) and we place \( (\nu, \mu) \in \prec \) otherwise.

\( \square \)

Suppose \( T = (N, o, p) \) is a tree. We say \( \prec \) is a tree ordering of \( T \) if

(i) \( \prec \) is well ordering of \( N \);
(ii) \( o \prec \nu \) for all \( \nu \in N \sim \{o\} \);
(iii) if \( \mu, \nu \in N \sim \{o\} \) and \( p(\mu) \prec p(\nu) \) then \( \mu \prec \nu \).

Suppose \( \prec \) is a tree ordering of \( T \).

Proposition 1.3. Suppose \( \mu, \nu \in N \sim \{o\} \) and \( p(\mu) \prec p(\nu) \). Then \( \mu \prec \nu \).
Suppose for each \(2\) then there is one and only one tree ordering \(\prec\). (I)

Theorem 1.3.

Corollary 1.2. Suppose \(\mu \in N \sim \{0\}\). Then \(p(\mu) \prec \mu\).

Theorem 1.2. Suppose for each \(\xi \in N\) we are given a well ordering \(\prec_{\xi}\) of \(c(\xi)\). Then there is a one and only one tree ordering \(\prec\) of \(T\) such that

\[
\mu \prec \nu \iff \mu \prec_{\xi} \nu \quad \text{for} \ \xi \in N \quad \text{and} \quad \mu, \nu \in c(\xi).
\]

Proof. Let \(D = \max\{d(\mu) : \mu \in N\}\) and for each \(d \in I[0, D]\) let \(N_d = \{\mu \in N : d(\mu) \leq d\}\) and let \(N_{[d]} = \bigcup_{0 \leq d \leq D} N_d\). For each \(d \in I[0, D]\) we define the relation \(W_d\) on \(N_{[d]}\) by induction on \(d\) as follows. We let \(W_0 = \emptyset\). If \(d \in I[0, D]\) we let

\[
W_{d+1} = W_d \cup W_d^1 \cup W_d^2 \cup W_d^3
\]

where

\[
W_d^1 = N_{[d]} \times N_{d+1}; \\
W_d^2 = \{(\mu, \nu) \in N_{d+1} \times N_{d+1} : (p(\mu), p(\nu)) \in W_d\}; \\
W_d^3 = \{(\mu, \nu) \in N_{d+1} \times N_{d+1} : \text{for some} \ \xi, \ \xi \in N, p(\mu) = \xi = p(\nu) \quad \text{and} \quad \mu \prec_{\xi} \nu\}.
\]

One verifies by induction on \(d \in I[0, D]\) that \(W_d\) is a well ordering of \(N_{[d]}\). □

Theorem 1.1. Suppose for each \(\mu \in N\) we are given a well ordering \(\prec_{\mu}\) of \(c(\mu)\). Then there is one and only one tree ordering \(\prec\) of \(T\) such that, for each \(\xi \in N\) and each \(\mu, \nu \in c(\xi)\),

\[
(2) \quad \mu \prec_{\xi} \nu \iff \mu \prec \nu.
\]

Proof. We define a relation \(\prec\) on \(N\) as follows. Suppose \(\mu, \nu \in N\) and \(\mu \neq \nu\). If \(\mu \in A(\nu)\) then \(\mu \prec \nu\). If \(\nu \in A(\mu)\) then \(\nu \prec \mu\).

Suppose \(\mu \not\in A(\nu)\) and \(\nu \not\in A(\mu)\). Let \(\xi\) be the deepest common ancestor of \(\mu\) and \(\nu\). Let \(m, n \in N^+\) be such that \(p^m(\mu) = \xi = p^n(\nu)\). Then \(\{p^{m-1}(\mu), p^{n-1}(\nu)\} \subset c(\xi)\). Then \(\mu \prec \nu\) if \(p^{m-1} \prec_{\xi} p^{n-1}(\nu)\) and \(\nu \prec \mu\) if \(p^{n-1}(\nu) \prec_{\xi} p^{m-1}(\xi)\).

We leave to the reader the straightforward proof that \(\prec\) is a tree ordering of \(T\) as well as the straightforward verification using ?? and ?? that it is the only ordering of \(T\) satisfying (??). □

Suppose \(O_i = (N_i, o_i, p_i, \prec_i), i = 1, 2\) are ordered trees and \(i : N_1 \rightarrow N_2\). We say \(i\) is an **isomorphism from** \(O_1\) **to** \(O_2\) if \(i\) is an isomorphism from \((N_1, o_1, p_1)\) to \((N_2, o_2, p_2)\) and

\[
\mu, \nu \in N_1 \text{ and } \mu \prec_1 \nu \Rightarrow i(\mu) \prec_2 i(\nu).
\]

Suppose \(O = (N, o, p, \prec)\) is an ordered tree and \(U = (O, \sigma, q)\) is a subtree of \(T = (N, o, p)\). Let

\[
\prec =
\]
\{(\mu, \nu) \in \mathcal{O} \times \mathcal{O} : \mu \prec \nu\} and note that \((\mathcal{O}, \sigma, q, \prec)\) is an ordered tree which we call the ordered tree associated to \(\mathcal{U}\). In particular, if \(\nu \in \mathcal{N}'\), \(\mathcal{O} = \mathcal{N}', \sigma = \nu\) and \(q = q_0\) we call this ordered tree the ordered tree associated to \(\nu\).

1.5. **Tree codes.** Suppose \(N \in \mathbb{N}^+\). We let 

\[ T(N) \]

be the set of functions \(P : I[1, N) \to I[0, N)\) such that 

\[ P^{-1}\{0\} \neq \emptyset; \]

\[ i \in I[1, N) \Rightarrow P(i) < i; \]

\[ i, j \in I(1, N) \text{ and } i < j \Rightarrow P(i) \leq P(j). \]

**Proposition 1.5.** Suppose \(N \in \mathbb{N}^+\) and \(P \in T(N)\). Then \(\mathcal{T} = (I[1, N), P, 0)\) is a tree and the standard ordering of \(I[1, N)\) is a tree ordering of \(\mathcal{T}\).

**Proof.** \[ \square \]

Suppose \(\mathcal{T} = (\mathcal{N}, o, p)\) is a tree and \(\prec\) is tree order for \(\mathcal{T}\). Let \(N = \text{card} \mathcal{N}\) and let 

\[ W : I[0, N) \to \mathcal{N} \]

be such that 

\[ i, j \in I[0, N) \text{ and } i < j \Rightarrow W(i) \prec W(j). \]

Note that \(W(0) = o\) and \(W\) is univalent; in particular, \(W^{-1}\) is a function. Let 

\[ P : I[1, N) \to \mathcal{N} \]

be such that 

\[ P(i) = W^{-1}(p(W(i)) \quad \text{for } i \in I[1, N). \]

We call \(P\) the tree code for \((\mathcal{T}, \prec)\).

**Proposition 1.6.** \(P \in T(N)\).

**Proof.** Suppose \(i \in I[1, N)\). Let \(\mu = W(i)\). Then \(W(P(i)) = p(W(i)) = p(\mu) \prec \mu = W(i)\) so \(P(i) < i\).

Suppose \(i, j \in I(1, N), i < j\) and \(P(i) \neq P(j)\). Let \(\mu = W(i)\) and let \(\nu = W(j)\). Then \(\mu \prec \nu\). We have \(W(P(i)) = p(W(i)) = p(\mu)\) and \(W(P(j)) = p(W(j)) = p(\nu)\). Since \(P(i) \neq P(j)\) we have \(\mu \neq \nu\) so \(p(\mu) < p(\nu)\) so \(P(i) < P(j)\). \[ \square \]

**Theorem 1.4.** Suppose, for \(i = 1, 2\), \(\mathcal{T}_i = (\mathcal{N}_i, p_i, \alpha_i)\) is a tree and \(\prec_i\) is a tree ordering of \(\mathcal{T}_i\). Then \(\mathcal{T}_i, i = 1, 2,\) are \((\prec_1, \prec_2)\) isomorphic if and only if they have the same tree codes.

**Proof.** Let \(\iota : \mathcal{N}_1 \to \mathcal{N}_2\) be a \((\prec_1, \prec_2)\) isomorphism and let \(T_i, i = 1, 2,\) be the \(\prec_i\) tree codes of \(\mathcal{T}_i\) respectively. Then

\[ P_2 = W_2^{-1} \circ p_2 \circ W_2 \]

\[ = W_2^{-1} \circ \iota \circ p_1 \circ \iota^{-1} \circ W_2 \]

\[ = W_2^{-1} \circ \iota \circ W_1 \circ P_1 \circ W_1^{-1} \circ \iota^{-1} \circ W_2 \]

\[ = (W_2^{-1} \circ \iota \circ W_1) \circ P_1 \circ (W_1^{-1} \circ \iota^{-1} \circ W_2); \]

Since \(W_2^{-1} \circ \iota \circ W_1\) and \(W_2^{-1} \circ \iota^{-1} \circ W_2\) are order preserving maps of \(I[1, N)\) to itself they are equal to the identity map of \(I[1, N)\).

\[ \square \]
1.6. Growing trees. Suppose

\[ T = (\mathcal{N}, o, p) \]

is a tree and

\[ O \quad \text{and} \quad P \]

satisfy the following conditions:

(i) \( O \) and \( P \) are functions with domain the leaf nodes of \( T \);
(ii) for each \( \nu \in \mathcal{N} \), \( (O(\nu), \nu, P(\nu)) \) is a tree;
(iii) the family

\[ \{i(T)\} \cup \{O(\nu) : \nu \in I(T)\} \]

is disjointed.

Let

\[ U = i(T) \cup \left( \bigcup \{O(\nu) : \nu \in I(T)\} \right) \]

and let

\[ q = p \cup \left( \bigcup \{P(\nu) : \nu \in I(T)\} \right). \]

We leave it as an exercise for the reader to verify that

\[ (U, o, q) \]

is a tree; that \( \mathcal{N} \subset U \); and that, for each \( \nu \in I(T) \), \( (O(\nu), \nu, P(\nu)) \) is the subtree associated to the node \( \nu \) of \( U \).

Now let us suppose that

\[ (\mathcal{N}, o, p, <) \]

is an ordered tree and that, for each \( \nu \in I(T) \), \( <_\nu \) is such that

\[ (O(\nu), \nu, P(\nu), <_\nu) \]

is an ordered tree. We leave it to the reader to verify that there is one and only one

\(<\)

such that

(i) \( (U, o, q, <) \) is an ordered tree;
(ii) if \( \mu, \xi \in \mathcal{N} \) and \( \mu < \xi \) then \( \mu < \xi \);
(iii) if \( \nu \in I(T) \), \( \mu, \xi \in O(\nu) \) and \( \mu <_\nu \xi \) then \( \mu < \xi \).

2. Context free grammars.

Definition 2.1. By a context free grammar ordered triple

\[ G = (T, N, s, P) \]

such that

(i) \( T \) is a set;
(ii) \( N \) is a set, \( T \cap N = \emptyset \) and \( s \in N \);
(iii) \( P \subset N \times (T \cup N)^* \);
The members of $T$ are called tokens or terminal symbols. The members of $N$ are called nonterminals or nonterminal symbols. $s$ is called the start symbol. The members of $P$ are called productions. Instead of writing $(r, s) \in P$ one often writes
\[ r := \epsilon \quad \text{if} \quad |s| = 0 \]
and
\[ r := s_0 \ s_1 \ \cdots \ s_{|s|-1} \quad \text{if} \quad |s| > 0. \]
If $r \in N$, $n \in \mathbb{N}^+$ and $s_0, \ldots, s_{n-1} \in (T \cup N)^*$ one often writes
\[ r := s_0 \mid s_1 \mid \cdots \mid s_{n-1} \]
instead of
\[ (r, s_i) \in P, \quad i \in I(n). \]
Obviously, if $(T, N, s, P)$ is a context free grammar then so is $(T, N, t, P)$ if $t \in N$.

**Definition 2.2.** A parse tree $Q$ for the context free grammar $G = (T, N, s, P)$ is an ordered quintuple
\[ (N, o, p, <, f) \]
such that
(i) $(N, o, p, <)$ is an ordered tree;
(ii) $f: N \to T \cup N$;
   (a) if $\nu \in i(T)$ then $f(\nu) \in N$;
   (b) if $\nu \in i(T)$ then $f(\nu) \in T$;
(iii) if $\nu \in i(T)$ and $\nu$ has $m$ children
\[ \mu_0 < \mu_2 < \ldots < \mu_{m-1} \]
then
\[ f(\nu), (f(\mu_0), \ldots, f(\mu_{m-1}))) \in P. \]
Notice that the notion of parse tree is independent of $s$.
We define
\[ < Q > \in (T)^* \]
as follows. Let
\[ \mathcal{L} = \{ \nu \in i(T) : f(\nu) \neq \epsilon \} \]
and let
\[ n = |\mathcal{L}|. \]
If $n = 0$ we let
\[ < Q > = \epsilon \]
and if $n > 0$ and
\[ \nu_0 < \nu_1 < \cdots < \nu_{n-1} \]
are the members of $\mathcal{L}$ we let
\[ < Q > = (f(\nu_0))|f(\nu_1)|\cdots|f(\nu_{n-1})|. \]
For each $t \in N$ we let
\[ L(G, t) \]
be the set of $< Q >$ as above where $f(o) = t$. We let
\[ L(G) = L(G, s) \]
and we call this language on the alphabet $T$ the language generated by $G$. 
Definition 2.3. We say the context free grammar $G$ is **good** if $Q_i = (N_i, o_i, p_i, <_i, f_i), i = 1, 2$, are parse trees such that $<_1 Q_1 = _2 Q_2$ then the ordered trees $(N_i, o_i, p_i, <_i)$ are isomorphic; recall that this is the case if and only if they have the same tree codes. We say $G$ is **bad** if it is not good.

2.1. A bad grammar. Let $B$ be the context free grammar defined as follows. Let

$$T = N \cup \{-, +, *\}$$

and let

$$N = \{\text{expr}\}.$$ 

Let $\text{expr}$ be the start symbol. Let the productions be given by

$$\text{expr} := j \quad \text{for } j \in \mathbb{N}$$

$$\text{expr} := - \text{expr}$$

$$\text{expr} := \text{expr} + \text{expr}$$

$$\text{expr} := \text{expr} * \text{expr}$$

Note that there are an infinite number of productions.

This grammar derives the string $27 + 5 * 298$ with two nonisomorphic parse trees so it is bad.

There are a number of ways to deal with this problem. One is to introduce parentheses which we now do. As we shall see, there are other ways to deal with this problem.