1. Trees; context free grammars.

1.1. **Trees.**

Definition 1.1. *By a* **rooted tree** *we mean an ordered triple*

$$\mathcal{T} = (\mathcal{N}, \mathrm{o}, p)$$

such that

(i) *$\mathcal{N}$ is a set;*
(ii) *$\mathrm{o} \in \mathcal{N}$;*
(iii) *$p : \mathcal{N} \sim \{\mathrm{o}\} \to \mathcal{N}$;*
(iv) *if $\nu \in \mathcal{N} \sim \{\mathrm{o}\}$ there is $d \in \mathbb{N}^+$ such that $(\nu, \mathrm{o}) \in p^d$.*

One calls o *the* **root of** *$\mathcal{T}$.*

The members of $\mathcal{N}$ are called **nodes**. o is called the **root node**. If $\nu \in \mathcal{N} \sim \{\mathrm{o}\}$ we call $p(\nu)$ the **parent of** ν. If $\nu \in \mathcal{N}$ we let

$$\mathbf{c}(\nu) = p^{-1}[\{\nu\}] = \{\mu \in \mathcal{N} \sim \{\mathrm{o}\} : p(\mu) = \nu\}$$

and call the members of this set the **children of** ν. A node which has children is called an **interior node**. A node which has no children is called a **leaf node**. We let

$$\mathbf{i} \quad \text{and} \quad \mathbf{l}$$

be the set of interior nodes of $\mathcal{T}$ and the set of leaf nodes of $\mathcal{T}$, respectively.

Lemma 1.1. *Suppose $n \in \mathbb{N}^+$. Then $p^{-n}[\{\mathrm{o}\}] \subset \mathcal{N} \sim \{\mathrm{o}\}$.*

Proof. Suppose, contrary to the Lemma, $\mathrm{o} \in p^{-n}[\{\mathrm{o}\}]$. Since $p^{-n} = (p^n)^{-1}$ we have $(\mathrm{o}, \mathrm{o}) \in p^n$ so

$$\mathrm{o} \in \mathbf{dmn}\, p^n \begin{cases} = \{(\nu, \nu) : \nu \in \mathcal{N} \sim \{\mathrm{o}\}\} & \text{if } n = 0, \\ \subset \mathbf{dmn}\, p & \text{if } n > 0 \end{cases}$$

which is incompatible with 1.1(iii). □

Lemma 1.2. *Suppose $m, n \in \mathbb{N}$ and $p^{-m}[\{\mathrm{o}\}] \cap p^{-n}[\{\mathrm{o}\}] \neq \emptyset$. Then $m = n$.*

Proof. Suppose, contrary to the Lemma, $\mu \in p^{-m}[\{\mathrm{o}\}] \cap p^{-n}[\{\mathrm{o}\}]$ and $m > n$. Since $p^{-n} = (p^n)^{-1}$ we have $(\mathrm{o}, \mu) \in p^{-n}$; also, $(\mu, \mathrm{o}) \in p^m$ so that

$$(\mathrm{o}, \mathrm{o}) \in p^{-n} \circ p^m = p^{m-n};$$

which is incompatible with 1.1(iii). □

Definition 1.2. *Let*

$$\mathbf{d} = \{(\mathrm{o}, 0)\} \cup \left(\bigcup_{n \in \mathbb{N}^+} p^{-n}[\{o\}] \times \{n\} \right).$$

here **d** *stands for "depth".*

Obviously,

$$p^{-n}[\{o\}] = \mathbf{d}^{-1}[\{n\}] \quad \text{for } n \in \mathbb{N}^+. \tag{1}$$

Theorem 1.1. **d** *is a function with domain $\mathcal{N}$.*

Proof. $o \in \mathbf{dmn}\,\mathbf{d}$ since $(o,0) \in \mathbf{d}$. If $\nu \in \mathcal{N} \sim \{o\}$ there is by (iv) $n \in \mathbb{N}^+$ such that $(\nu, o) \in p^n$ so $\nu \in p^{-n}[\{o\}] \subset \mathbf{dmn}\,\mathbf{d}$.

Suppose, contrary to the Theorem, there are $\nu \in \mathcal{N}$ and n_i, $i = 1, 2$, such that $n_2 > n_1$ and $(\nu, n_i) \in \mathbf{d}$, $i = 1, 2$.

In case $n_1 = 0$ we would have $\nu = o$ and $\nu \in p^{-n_2}[\{o\}]$ which is excluded by Lemma ??.

In case $n_1 > 0$ we would have $\nu \in p^{-n_i}[\{o\}]$, $i = 1, 2$. Then $(\nu, o) \in p^{-n_1}$ and $(o, \nu) \in (p^{-n_2})^{-1} = p^{n_2}$ so $(o, o) \in p^{-n_1} \circ p^{n_2} = p^{n_2 - n_1}$. Thus $o \in \mathbf{dmn}\, p^{n_2 - n_1} \subset \mathbf{rng}\, p$ which is incompatible with Lemma 1.1(iii). □

Corollary 1.1. *$\{p^{-d}[\{o\}] : d \in \mathbb{N}\}$ is a partition of $\mathcal{N}$.*

For each $\mu \in \mathcal{N}$ we let

$$\mathbf{A}(\mu) = \bigcup_{n \in \mathbb{N}^+} \{\nu : (\mu, \nu) \in p^n\}$$

and we let

$$\mathbf{D}(\mu) = \bigcup_{n \in \mathbb{N}^+} \{\nu : (\nu, \mu) \in p^n\};$$

The members of $\mathbf{A}(\mu)$ are called **ancestors of** μ and the members of $\mathbf{D}(\mu)$ are called **descendants of** μ.

Proposition 1.1. $\mathbf{A}(o) = \emptyset$.

Proof. Suppose $n \in \mathbb{N}^+$, $(o, \nu) \in p^n$. Then $o \in \mathbf{dmn}\, p^n \subset \mathbf{dmn}\, p = \mathcal{N} \sim \{o\}$ which is impossible. □

Proposition 1.2. *Suppose $\mu \in \mathcal{N} \sim \{o\}$. Then $\mu \in \mathbf{dmn}\,\mathbf{d}$, $\mathbf{d}(\mu) > 0$, $\mu \in p^{-\mathbf{d}(\mu)}[\{o\}]$,*

$$\mathbf{A}(\mu) = \{p^n(\mu) : n \in \mathbb{I}(1, \mathbf{d}(\mu)]\}$$

and $\mathbf{d}|\mathbf{A}(\mu)$ is a univalent function with range $\mathbb{I}[0, \mathbf{d}(\mu))$.

Proof. $\mu \in \mathbf{dmn}\,\mathbf{d}$ by Theorem ??. Since $\mu \neq o$ we have $\mathbf{d}(\mu) > 0$ and $\mu \in p^{-\mathbf{d}(\mu)}[\{o\}]$; in particular, $(\mu, o) \in p^{\mathbf{d}(\mu)}$ and $(o, \mu) \in p^{\mathbf{d}(\mu)}$. Suppose $n \in \mathbb{N}^+$ and $(\mu, \nu) \in p^n$. Then, as $(\nu, mu) \in p^{-n}$ we have $(\nu, o) \in p^{-n} \circ p^{\mathbf{d}(\mu)} = p^{\mathbf{d}(\mu) - n}$ so $\nu \in p^{n - \mathbf{d}\mu}[\{o\}]$. Since $o \notin \mathbf{dmn}\, p$ we find that $n - \mathbf{d}(\mu) < 0$ and $\mathbf{d}(\nu) = \mathbf{d}(\mu) - n \in \mathbb{I}[0, \mathbf{d}(\mu))$. □

1.2. **Subtrees.**

Definition 1.3. *We say the rooted tree $\mathcal{U} = (\mathcal{O}, \sigma, q)$ is a* **rooted subtree** *of $\mathcal{T} = (\mathcal{N}, o, p)$ if $\sigma \in \mathcal{O} \subset \mathcal{N}$ and $q = p|(\mathcal{O} \sim \{\sigma\})$.*

Given $\nu \in \mathcal{N}$ one easily verifies that

$$(\{\nu\} \cup \mathbf{D}(\nu), \nu, p|\mathbf{D}(\nu))$$

is a rooted subtree of $\mathcal{T}$ which we call the **rooted subtree associated to the node ν.**

1.3. **Isomorphisms.** Suppose $\mathcal{T}_i = (\mathcal{N}_i, o_i, p_i)$, $i = 1, 2$ are rooted trees and $\iota : \mathcal{N}_1 \to \mathcal{N}_2$. We say ι is an **isomorphism from $\mathcal{T}_1$ to $\mathcal{T}_2$** if ι is univalent, $\mathbf{rng}\,\iota = \mathcal{N}_2$ and

$$p_2 \circ \iota = \iota \circ p_1;$$

it follows that $\iota(o_1) = o_2$ and that ι^{-1} is an isomorphism from $\mathcal{T}_2$ to $\mathcal{T}_1$.

1.4. **Tree orderings.** Suppose $\mathcal{T} = (\mathcal{N}, \mathrm{o}, p)$ is a tree. We say $\prec$ is a **tree ordering of** $\mathcal{T}$ if

(i) $\prec$ is well ordering of $\mathcal{N}$;
(ii) if $\mu, \nu \in \mathcal{N}$ and $\mathbf{d}(\mu) < \mathbf{d}(\nu)$ then $\mu \prec \nu$.
(iii) if $\mu, \nu \in \mathcal{N} \sim \{\mathrm{o}\}$, $\mathbf{d}(\mu) = \mathbf{d}(\nu)$ and $p(\mu) \prec p(\nu)$ then $\mu \prec \nu$.

Proposition 1.3. *Suppose* $\mathbf{W}$ *is a function with domain* $\mathbb{N}$ *whose value at* $n \in \mathbb{N}$ *is a well ordering of* $\mathbf{d}^{-1}[\{n\}]$ *such that*

$$(p(\mu), p(\nu)) \in \mathbf{W}(n) \;\Rightarrow\; (\mu, \nu) \in \mathbf{W}(n+1)$$

whenever $n \in \mathbb{N}$ *and* $\mu, \nu \in \mathbf{d}^{-1}[\{n+1\}]$. *Then*

$$\left(\bigcup_{n=0}^{\infty} \mathbf{W}(n) \right) \cup \{(\mu, \nu) \in \mathcal{N} \times \mathcal{N} : \mathbf{d}(\mu) < \mathbf{d}(\nu)\}$$

is a tree ordering of $\mathcal{T}$.

Theorem 1.2. *Suppose for each* $\xi \in \mathcal{N}$ *we are given a well ordering* $\prec_\xi$ *of* $\mathbf{c}(\xi)$. *Then there is one and only one tree ordering* $\prec$ *of* $\mathcal{T}$ *such that, for each* $\xi \in \mathcal{N}$ *and each* $\mu, \nu \in \mathbf{c}(\xi)$,

$$\mu \prec_\xi \nu \;\Leftrightarrow\; \mu \prec \nu. \tag{2}$$

Proof. With regard to existence, we begin by constructing by induction on $n \in \mathbb{N}$ a well ordering W_n of $\mathbf{d}^{-1}[\{n\}]$ as follows. We let $W_0 = \emptyset$. If $n \in \mathbb{N}$ and W_n is constructed We let

$$U_{n+1} = \{(\mu, \nu) \in \mathbf{d}^{-1}[\{n+1\}] : \mu \neq \nu \text{ and} (p(\mu), p(\nu) \in W_n\};$$
$$V_{n+1} = \{(\mu, \nu) \in \mathbf{d}^{-1}[\{n+1\}] : \mu \neq \nu, p(\mu) = p(\nu) \text{ and } (\mu, \nu) \in \prec_{p(\mu)}\};$$

and we let $W_{n+1} = U_{n+1} \cup V_{n+1}$. We let

$$\prec = \{(\mu, \nu) \in \mathcal{N} \times \mathcal{N} : \mathbf{d}(\mu) < \mathbf{d}(\nu)\} \cup \left(\bigcup_{n \in \mathbb{N}} W_n \right).$$

It should be clear that $\prec$ is the unique tree ordering of $\mathcal{T}$ such that (2) holds. □

Suppose $\mathcal{O}_i = (\mathcal{N}_i, \mathrm{o}_i, p_i, \prec_i)$, $i = 1, 2$ are ordered trees and $\iota : \mathcal{N}_1 \to \mathcal{N}_2$. We say ι is an **isomorphism from** $\mathcal{O}_1$ **to** $\mathcal{O}_2$ if ι is an isomorphism from $(\mathcal{N}_1, \mathrm{o}_1, p_1)$ to $(\mathcal{N}_2, \mathrm{o}_2, p_2)$ and

$$\mu, \nu \in \mathcal{N}_1 \text{ and } \mu \prec_1 \nu \;\Rightarrow\; \iota(\mu) \prec_2 \iota(\nu).$$

Suppose $\mathcal{O} = (\mathcal{N}, \mathrm{o}, p, \prec)$ is an ordered tree and $\mathcal{U} = (\mathcal{O}, \sigma, q)$ is a subtree of $\mathcal{T} = (\mathcal{N}, \mathrm{o}, p)$. Let

$$\prec =$$

$\{(\mu, \nu) \in \mathcal{O} \times \mathcal{O} : \mu \prec \nu\}$ and note that $(\mathcal{O}, \sigma, q, \prec)$ is an ordered tree which we call the **ordered tree associated to** $\mathcal{U}$. In particular, if $\nu \in \mathcal{N}$, $\mathcal{O} = \mathcal{N}_\nu$, $\sigma = \nu$ and $q = q_\nu$ we call this ordered tree the **ordered tree associated to** ν.

1.5. **Tree codes.** Suppose I is an initial segment of $\mathbb{N}$. We let

$$\mathbf{T}(I)$$

be the set of functions $P : I \sim \{0\} \to I$ such that

$$P^{-1}[\{0\}] \neq \emptyset; \tag{3}$$

$$i \in I \sim \{0\} \;\Rightarrow\; P(i) < i; \tag{4}$$

$$i, j \in I \sim \{0\} \text{ and } i < j \;\Rightarrow\; P(i) \leq P(j). \tag{5}$$

Proposition 1.4. *Suppose I is an intial segment of $\mathbb{N}$ and $P \in \mathbf{T}(I)$. Then $\mathcal{T} = ((I, 0, P), P, 0)$ is a tree and the standard well ordering of I is a tree ordering of $\mathcal{T}$.*

Proof. □

Suppose $\mathcal{T} = (\mathcal{N}, \mathrm{o}, p)$ is a tree and $\prec$ is tree ordering of $\mathcal{T}$. Let $N = \mathbf{card}\,\mathcal{N}$ and let

$$W : \mathbf{I}[0, N) \to \mathcal{N}$$

be such that

$$i, j \in \mathbf{I}[0, N) \text{ and } i < j \;\Rightarrow\; W(i) \prec W(j).$$

Note that $W(0) = \mathrm{o}$ and W is univalent; in particular, W^{-1} is a function. Let

$$P = W^{-1} \circ p \circ W.$$

Proposition 1.5. $P \in \mathbf{T}(N)$.

Remark 1.1. *We call P the* **tree code for** $(\mathcal{T}, \prec)$**.**

Proof. Suppose $i \in \mathbb{I}(1, N)$. Let $\mu = W(i)$. Then $W(P(i)) = p(W(i)) = p(\mu) \prec \mu = W(i)$ so $P(i) < i$.

Suppose $i, j \in \mathbb{I}(1, N)$, $i < j$ and $P(i) \neq P(j)$. Let $\mu = W(i)$ and let $\nu = W(j)$. Then $\mu \prec \nu$. We have $W(P(i)) = p(W(i)) = p(\mu)$ and $W(P(j)) = p(W(j)) = p(\nu)$. Since $P(i) \neq P(j)$ we have $\mu \neq \nu$ so $p(\mu) < p(\nu)$ so $P(i) < P(j)$. □

Theorem 1.3. *Suppose, for $i = 1, 2$, $\mathbf{T}_i = (\mathcal{N}_i, p_i, \mathrm{o}_i)$ is a tree and $\prec_i$ is a tree ordering of $\mathbf{T}_i$. Then $\mathbf{T}_i$, $i = 1, 2$, are $(\prec_1, \prec_2)$ isomorphic if and only if they have the same tree codes.*

Proof. Let $\iota : \mathcal{N}_1 \to \mathcal{N}_2$ be a $(\prec_1, \prec_2)$ isomorphism and let T_i, $i = 1, 2$, be the $\prec_i$ tree codes of $\mathcal{T}_i$ respectively. Then

$$\begin{aligned}
P_2 &= W_2^{-1} \circ p_2 \circ W_2 \\
&= W_2^{-1} \circ \iota \circ p_1 \circ \iota^{-1} \circ W_2 \\
&= W_2^{-1} \circ \iota \circ W_1 \circ P_1 \circ W_1^{-1} \circ \iota^{-1} \circ W_2 \\
&= (W_2^{-1} \circ \iota \circ W_1) \circ P_1 \circ (W_1^{-1} \circ \iota^{-1} \circ W_2);
\end{aligned}$$

Since $W_2^{-1} \circ \iota \circ W_1$ and $W_1^{-1} \circ \iota^{-1} \circ W_2$ are order preserving maps of $\mathbf{I}[1, N)$ to itself they are equal to the identity map of $\mathbf{I}[1, N)$. □

1.6. **Growing trees.** Suppose

$$\mathcal{T} = (\mathcal{N}, \mathrm{o}, p)$$

is a tree and

$$\mathcal{O} \quad \text{and} \quad \mathcal{P}$$

satisfy the following conditions:

(i) $\mathcal{O}$ and $\mathcal{P}$ are functions with domain the leaf nodes of $\mathcal{T}$;
(ii) for each $\nu \in \mathbf{l}(\mathcal{T})$, $(\mathcal{O}(\nu), \nu, \mathcal{P}(\nu))$ is a tree;
(iii) the family

$$\{\mathbf{i}(\mathcal{T})\} \cup \{\mathcal{O}(\nu) : \nu \in \mathbf{l}(\mathcal{T})\}$$

is disjointed.

Let

$$\mathcal{U} = \mathbf{i}(\mathcal{T}) \cup \left(\bigcup \{\mathcal{O}(\nu) : \nu \in \mathbf{l}(\mathcal{T})\}\right)$$

and let

$$q = p \cup \left(\bigcup \{\mathcal{P}(\nu) : \nu \in \mathbf{l}(\mathcal{T})\}\right).$$

We leave it as an exercise for the reader to verify that

$$(\mathcal{U}, \mathrm{o}, q)$$

is a tree; that $\mathcal{N} \subset \mathcal{U}$; and that, for each $\nu \in \mathbf{l}(\mathcal{T})$, $(\mathcal{O}(\nu), \nu, \mathcal{P}(\nu))$ is the subtree associated to the node ν of $\mathcal{U}$.

Now let us suppose that

$$(\mathcal{N}, \mathrm{o}, p, <)$$

is an ordered tree and that, for each $\nu \in \mathbf{l}(\mathcal{T})$, $<_\nu$ is such that

$$(\mathcal{O}(\nu), \nu, \mathcal{P}(\nu), <_\nu)$$

is an ordered tree. We leave it to the reader to verify that there is one and only one

$$\prec$$

such that

(i) $(\mathcal{U}, \mathrm{o}, q, \prec)$ is an ordered tree;
(ii) if $\mu, \xi \in \mathcal{N}$ and $\mu < \xi$ then $\mu \prec \xi$;
(iii) if $\nu \in \mathbf{l}(\mathcal{T})$, $\mu, \xi \in \mathcal{O}(\nu)$ and $\mu <_\nu \xi$ then $\mu \prec \xi$.

2. Context free grammars.

For a set A we let $A^0 = \{\mathbf{null}\}$ and we let

$$A^* = \bigcup_{n=0}^{\infty} A^n$$

. For $a \in A^*$ we let

$$|a| \in \mathbb{N}$$

be such that $|a| = n$ if $a \in A^n$.

If $a, b \in A^*$ we define the **concatenation**

$$a\,b \in A^*$$

by requiring that

$$a\,b = b \quad \text{if } a = \mathbf{null}; \quad a\,b = a \quad \text{if } b = \mathbf{null};$$

and that

$$(a\,b)_i = \begin{cases} a_i & \text{if } i \in \mathbb{I}[1, M], \\ b_{i-M} & \text{if } i \in \mathbb{I}(M, M+N]. \end{cases}$$

Obviously,

$$|a\,b| = |a||b|.$$

Definition 2.1. *By a* **context free grammar** *ordered triple*

$$\mathcal{G} = (T, N, \mathcal{P})$$

such that

(i) *T is a set, N is a set and $T \cap N = \emptyset$;*
(iii) *$\mathcal{P} \subset N \times (T \cup N)^*$;*

The members of T are called **tokens** or **terminal symbols**. The members of N are called **nonterminals** or **nonterminal symbols**. The members of $\mathcal{P}$ are called **productions**. For $A, B \in (T \cup N)^*$ we write

$$A \Rightarrow_{\mathcal{G}} B$$

if there are a production $(\mathbf{n}, \mathbf{s}) \in \mathcal{P}$ and $A_L, A_R, B_L, B_R \in (T \cup N)^*$ such that

$$A = A_L\,\mathbf{n}\,A_R \quad \text{and} \quad B = B_L\,\mathbf{s}\,B_R.$$

Suppose $\mathcal{T}_A = (\mathcal{N}_A, o_A, p_A)$ is a tree such that there is a univalent function L_A with domain $\mathbf{rng}\,A$ and range the leaf nodes of $\mathcal{T}_A$. We Let $\nu = L_A(\mathbf{n})$; let $\mathcal{N}_B = \mathcal{N}_A \cup (\mathbf{rng}\,\mathbf{s} \times \{\nu\})$; let $o_B = o_A$; let $p_B = p_A \cup ((\mathbf{rng}\,(\mathbf{s} \times \{\nu\})) \times \{\nu\})$; and let

$$L_B = (L_A|\mathcal{N}_A \sim \{\nu\}) \cup (\mathbf{rng}\,\mathbf{s} \times \{\nu\}).$$

We say the nonterminal $\mathbf{n}$ **generates** $\mathbf{t} \in T^*$ if there is a finite sequence $A_1 \ldots, A_N$ such that

$$\mathbf{n} = A_1 \Rightarrow_{\mathcal{G}} \cdots \Rightarrow_{\mathcal{G}} A_N = \mathbf{t}.$$

For each

Instead of writing $(r, s) \in \mathcal{P}$ one often writes

$$r := \epsilon \quad \text{if } |s| = 0$$

and

$$r := s_0\ s_1\ \cdots\ s_{|s|-1} \quad \text{if } |s| > 0.$$

If $r \in N$, $n \in \mathbb{N}^+$ and $s_0, \ldots, s_{n-1} \in (T \cup N)^*$ one often writes

$$r := s_0 \mid s_1 \mid \cdots \mid s_{n-1}$$

instead of

$$(r, s_i) \in \mathcal{P}, \quad i \in I(n).$$

Obviously, if $(T, N, \mathbf{s}, \mathcal{P})$ is a context free grammar then so is $(T, N, \mathbf{t}, \mathcal{P})$ if $\mathbf{t} \in N$.

Definition 2.2. *A* **parse tree $\mathcal{Q}$ for the context free grammar** $\mathcal{G} = (T, N, \mathbf{s}, \mathcal{P})$ *is an ordered quintuple*

$$(\mathcal{N}, \mathrm{o}, p, <, f)$$

such that

(i) *$(\mathcal{N}, \mathrm{o}, p, <)$ is an ordered tree;*
(ii) *$f : \mathcal{N} \to T \cup N$;*
  (a) *if $\nu \in \mathbf{i}(\mathcal{T})$ then $f(\nu) \in N$;*
  (b) *if $\nu \in \mathbf{l}(\mathcal{T})$ then $f(\nu) \in T$;*

(iii) *if* $\nu \in \mathbf{i}(\mathcal{T})$ *and* ν *has* m *children*

$$\mu_0 < \mu_2 < \ldots < \mu_{m-1}$$

then

$$(f(\nu), (f(\mu_0), \ldots, f(\mu_{m-1}))) \in \mathcal{P}.$$

Notice that the notion of parse tree is independent of $\mathbf{s}$.

We define

$$< \mathcal{Q} > \in (T)^*$$

as follows. Let

$$\mathcal{L} = \{\nu \in \mathbf{l}(\mathcal{T}) : f(\nu) \neq \epsilon\}$$

and let

$$n = |\mathcal{L}|.$$

If $n = 0$ *we let*

$$< \mathcal{Q} > = \epsilon$$

and if $n > 0$ *and*

$$\nu_0 < \nu_1 < \cdots < \nu_{n-1}$$

are the members of $\mathcal{L}$ *we let*

$$< \mathcal{Q} > = (f(\nu_0))|(f(\nu_1))| \cdots |(f(\nu_{n-1})).$$

For each $\mathbf{t} \in N$ *we let*

$$\mathbf{L}(\mathcal{G}, \mathbf{t})$$

be the set of $< \mathcal{Q} >$ *as above where* $f(\mathrm{o}) = \mathbf{t}$. *We let*

$$\mathbf{L}(\mathcal{G}) = \mathbf{L}(\mathcal{G}, \mathbf{s})$$

and we call this language on the alphabet T *the* **language generated by** $\mathcal{G}$.

Definition 2.3. *We say the context free grammar* $\mathcal{G}$ *is* **good** *if* $\mathcal{Q}_i = (\mathcal{N}_i, \mathrm{o}_i, p_i, <_i, f_i)$, $i = 1, 2$, *are parse trees such that* $< \mathcal{Q}_1 > = < \mathcal{Q}_2 >$ *then the ordered trees* $(\mathcal{N}_i, \mathrm{o}_i, p_i, <_i)$ *are isomorphic; recall that this is the case if and only if they have the same tree codes. We say* $\mathcal{G}$ *is* **bad** *if it is not good.*

2.1. **A bad grammar.** Let $\mathcal{B}$ be the context free grammar defined as follows. Let

$$T = \mathbb{N} \cup \{\, - \,,\, + \,,\, * \,\}$$

and let

$$N = \{\, \mathbf{expr} \,\}.$$

Let **expr** be the start symbol. Let the productions be given by

$$\mathbf{expr} := j \quad \text{for } j \in \mathbb{N}$$

$$\mathbf{expr} := - \;\; \mathbf{expr}$$

$$\mathbf{expr} := \mathbf{expr} \;\; + \;\; \mathbf{expr}$$

$$\mathbf{expr} := \mathbf{expr} \;\; * \;\; \mathbf{expr}$$

Note that there are an infinite number of productions.

This grammar derives the string $27 + 5 * 298$ with two nonisomorphic parse trees so it is bad.

There are a number of ways to deal with this problem. One is to introduce parentheses which we now do. As we shall see, there are other ways to deal with this problem.