1. The tough one.

We introduce the following notation. Suppose A is a statement and

$$\mathbf{free}(A) = \{x_{i_1}, \dots, x_{i_n}\}$$

where

$$i_1 < i_2 < \cdots < i_n$$

and suppose $t_1, \dots, t_n$ are terms. We let

$$\overline{A}(t_1, \dots, t_n) = A_{x_{i_1} \to t_1, \dots, x_{i_n} \to t_n}.$$

We will frequently write

$$x \text{ for } (x_1, \dots, x_n).$$

Let

$$V = \{x_i : i \in \{1, \dots, n\}.$$

Proposition 1.1. Suppose $N \in \mathbb{N}$ and $N > 4$. Let u be the term

$$(1 + (x_3 + 1) \cdot x_2)$$

and let Bt be the statement

$$\exists\, x_N\, (((x_1 = (u \cdot x_N) + x_4) \wedge (x_4 < u))$$

Then Bt strongly represents the Gödel β-function.

Proof. See p. 131 in Mendelson for the straightforward proof. □

Let us suppose that

(i) $g \in \mathbb{N}_1^n$ and g is represented by the statement A;
(ii) $h \in \mathbb{N}_1^{n+2}$ and h is represented by the statement B;
(iii) $f \in \mathbb{N}_1^{n+1}$,

$$f(x, 0) = g(x) \quad \text{for } x \in \mathbb{N}^n;$$
$$f(x, y+1) = h(x, y, f(y)) \quad \text{for } (x, y) \in \mathbb{N}^{n+1};$$

Let Bt be as above with N there greater than $n + 7$. Let C_1, C_2, C_3, C_4 be the statements

$$\exists\, x_{n+5}\, (\overline{Bt}(x_{n+3}, x_{n+4}, \overline{0}, x_{n+5}) \wedge \overline{A}(x, x_{n+5})$$
$$\overline{Bt}(x_{n+3}, x_{n+4}, x_{n+1}, x_{n+2})$$
$$\overline{Bt}(x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}) \wedge \overline{Bt}(x_{n+3}, x_{n+4}, Sx_{n+5}, x_{n+6}) \wedge \overline{B}(x, x_{n+5}, x_{n+6}, x_{n+7})$$
$$\forall\, x_{n+5}\, ((x_{n+5} < x_{n+1}) \to \exists\, x_{n+6}\, \exists\, x_{n+7}\, C_3)$$

respectively. Note that

$$\mathbf{free}(A) = V \cup \{x_{n+1}\};$$
$$\mathbf{free}(B) = V \cup \{x_{n+1}, x_{n+2}, x_{n+3}\};$$
$$\mathbf{free}(C_1) = V \cup \{x_{n+3}, x_{n+4}\};$$
$$\mathbf{free}(C_2) = \{x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}\};$$
$$\mathbf{free}(C_3) = V \cup \{x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}, x_{n+7}\};$$
$$\mathbf{free}(C_4) = V \cup \{x_{n+1}, x_{n+3}, x_{n+4}\};$$
$$\mathbf{free}(C_1 \wedge C_2 \wedge C_4) = V \cup \{x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}\};$$

Let C be the statement

$$\exists\, x_{n+3}\, \exists\, x_{n+4} (C_1 \wedge C_2 \wedge C_4).$$

Note that

$$\mathbf{free}(C) = V \cup \{x_{n+1}, x_{n+2}\}.$$

Theorem 1.1. f is represented by C.

Proof. Suppose $k \in \mathbb{N}^n$, $p \in \mathbb{N}$ and

$$f(k, i) = r_i, \quad 0 \le i \le p.$$

Let $\overline{k} = (\overline{k}_1, \dots, \overline{k}_n)$. We will show that

$$\vdash \overline{C}(\overline{k}, \overline{p}, \overline{r_p}) \tag{1}$$

Part One. Suppose $p = 0$.

Choose b, c such that $\beta(b, c, 0) = r_0$. Then

$$\vdash \overline{Bt}(\overline{b}, \overline{c}, \overline{0}, \overline{r_0}) \quad \text{which is} \quad \vdash \overline{C_2}(\overline{0}, \overline{r_0}, \overline{b}, \overline{c})$$

Since $g(k) = r_0$ we have

$$\vdash \overline{A}(\overline{k}, \overline{r_0})$$

thus

$$\vdash \exists x_{n+1}(\overline{Bt}(\overline{b}, \overline{c}, \overline{0}, x_{n+1}) \wedge \overline{A}(\overline{k}, x_{n+1})$$

which is

$$\vdash \overline{C_1}(\overline{k}, \overline{b}, \overline{c}). \tag{2}$$

Since $\vdash \sim (x_{n+5} < \overline{0})$ we have

$$\vdash (x_{n+5} < \overline{0}) \rightarrow \exists x_{n+6} \exists x_{n+7} \overline{C_3}(\overline{k}, \overline{b}, \overline{c}, x_{n+5}, x_{n+6}, x_{n+7})$$

Applying **Gen** we obtain

$$\vdash \overline{C_4}(\overline{k}, \overline{0}, \overline{b}, \overline{c}).$$

Thus

$$\vdash \overline{C_1 \wedge C_2 \wedge C_4}(\overline{k}, \overline{0}, \overline{r}, \overline{b}, \overline{c})$$

which implies that (1) holds if $p = 0$.

Part Two. Suppose $p > 0$. Choose $b, c \in \mathbb{N}$ such that

$$\beta(b, c, i) = r_i, \quad 0 \le i \le p.$$

This implies

$$\vdash \overline{Bt}(\overline{b}, \overline{c}, \overline{i}, \overline{r_i}) \quad \text{which implies} \quad \vdash \overline{C_2}(\overline{i}, \overline{r_i}, \overline{b}, \overline{c}) \quad \text{for } 0 \le i \le p. \tag{3}$$

Since

$$r_{i+1} = f(k, i+1) = h(k, i, f(k, i)) = h(k, i, r_i), \quad 0 \le i < p,$$

and $\overline{i+1} = S\overline{i}$ we have

$$\vdash \overline{Bt}(\overline{b}, \overline{c}, \overline{i}, \overline{r_i}) \wedge \overline{Bt}(\overline{b}, \overline{c}, S\overline{i}, \overline{r_i}) \wedge \overline{B}(\overline{k}, \overline{i}, \overline{r_i}, \overline{r_{i+1}}), \quad 0 \le i < p.$$

which is

$$\vdash \overline{C_3}(\overline{k}, \overline{b}, \overline{c}, \overline{i}, \overline{r_i}, \overline{r_{i+1}}), \quad 0 \le i < p.$$

This implies

$$\vdash \exists x_{n+6} \exists x_{n+7} \overline{C_3}(\overline{k}, \overline{b}, \overline{c}, \overline{i}, x_{n+6}, x_{n+7}), \quad 0 \le i < p$$

We infer from a straightforward Proposition (3.8(b') in Mendelson) (or, even better, see Lemma 5 on page 291 of Hodel) that

$$\vdash \overline{C_4}(\overline{k}, \overline{p}, \overline{b}, \overline{c}) \tag{4}$$

Taking the conjunction of (2), (3) and (4) we find that

$$\vdash \overline{C_1 \wedge C_2 \wedge C_4}(\overline{k}, \overline{p}, \overline{r_p}, \overline{b}, \overline{c})$$

which implies (1).

$\square$