1. Keep going...

We axiomatize propositional logic by using following rules of inference.

Suppose A, B, C are statements. Then

Ex $$\Big(\{A\}, (A \vee B)\Big)$$

Contr $$\Big(\{(A \vee A)\}, A\Big)$$

Ass $$\Big(\{A \vee (B \vee C)\}, (A \vee B) \vee C\Big)$$

EM $$\Big(\emptyset, \sim A \vee A\Big)$$

Cut $$\Big(\{(A \vee B), (\sim A \vee C))\}, (B \vee C)\Big)$$

are rules of inference.

Ex stands for "expansion"; **Contr** stands for "contraction"; **Ass** stands for "associative"; **EM** stands for "excluded middle"; and **Cut** stands for "cut".

An **axiom** is a rule of inference where the set of hypotheses is empty; thus **EM** is an axiom.

Theorem 1.1. OrComm

$$(\{A \vee B\}, B \vee A)\,.$$

Proof.

Hyp $$A \vee B$$

EM $$\sim A \vee A$$

Cut $$B \vee A$$

□

Theorem 1.2. OtherOrAss

$$(\{(A \vee B) \vee C\}, A \vee (B \vee C)).$$

Proof.

$$(A \vee B) \vee C$$
$$C \vee (A \vee B) \quad \textbf{OrComm}$$
$$(C \vee A) \vee B)) \quad \textbf{HalfAss}$$
$$B \vee (C \vee A) \quad \textbf{OrComm}$$
$$(B \vee C) \vee A \quad \textbf{HalfAss}$$
$$A \vee (B \vee C) \quad \textbf{OrComm}$$

□

Definition 1.1.

$$A \to B \equiv \sim A \vee B$$
$$A \wedge B \equiv \sim (\sim A \vee \sim B)$$
$$A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$$

Theorem 1.3. ModusPonens

$$(\{A, \sim A \vee B\}, B)\,.$$

Proof.

1.	A	**Hyp**
2.	$A \vee B$	**1, Ex**
3.	$\sim A \vee B$	**Hyp**
4.	$B \vee B$	**2, 3, Cut**
5.	B	**4, Contr**

□

Theorem 1.4.

$$(\{A, B\}, A \wedge B)$$

Proof.

$$A$$
$$B$$
$$\sim (A \wedge B) \vee (A \wedge B)$$
$$(\sim A \vee \sim B) \vee (A \wedge B) \quad \text{Not so fast!}$$
$$\sim A \vee (\sim B \vee (A \wedge B)$$
$$\sim B \vee (A \wedge B)$$
$$A \wedge B$$

□

Theorem 1.5.

$$(\{A\}, \sim\sim A)\,.$$

Proof.

Hyp	A
EM	$\sim\sim A \vee \sim A$
OrComm	$\sim A \vee \sim\sim A$
ModusPonens	$\sim\sim A$

□

Theorem 1.6.

$$(\{\sim\sim A\}, A)\,.$$

Proof.

Hyp	$\sim\sim A$
Ex	$\sim\sim A \vee A$
EM	$\sim A \vee A$
Cut	$A \vee A$
Contr	A

□

2. Maybe...

$$\{\sim B, \sim C\} \vdash \sim (B \vee C)$$

$\sim B$
$\sim C$
$(B \vee C) \vee \sim (B \vee C)$
$B(\vee C) \vee \sim (B \vee C)$
$C \vee \sim (B \vee C)$
$\sim (B \vee C)$

$$\{\sim A \vee C, \sim B \vee C, A \vee B\} \vdash C$$

$\sim A \vee C$
$\sim B \vee C$
$A \vee B$
$B \vee C$
C

$$\{A \to C, B \to C, A \vee B\} \vdash C$$

$\sim A \vee C$
$\sim B \vee C$
$A \vee B$
$B \vee C$
C

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Theorem 2.1. We have

$$A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$$

and

$$A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C).$$

Proof. Let $D = (A \wedge B) \vee (A \wedge C)$.

$$\begin{array}{l} A \wedge (B \vee C) \\ A \\ B \vee C \\ B \to B \\ B \to A \\ B \to A \wedge B \to D \\ C \to C \\ C \to A \\ C \to A \wedge C \to D \\ B \vee C \to D \\ D \end{array}$$

so

$$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C).$$

Replacing A, B, C by $\sim A, \sim B, \sim C$ we find that

$$\sim ((\sim A \wedge \sim B) \wedge (\sim A \wedge \sim C)) \to \sim (\sim A \wedge (\sim B \vee \sim C))$$

which implies that

$$(A \vee B) \wedge (A \vee B) \to A \vee (B \wedge C).$$

Let $D = (A \vee B) \wedge (A \vee C)$.

$$\begin{array}{l} A \vee (B \wedge C) \\ A \to A \vee B \\ A \to A \vee C \\ A \to D \\ B \wedge C \to B \to A \vee B \\ B \wedge C \to C \to A \vee C \\ B \wedge C \to D \\ A \vee (B \wedge C) \to D \\ D \end{array}$$

so

$$A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C).$$

Replacing A, B, C by $\sim A, \sim B, \sim C$ we find that

$$\sim ((\sim A \vee \sim B) \wedge (\sim A \vee \sim C)) \to \sim (\sim A \vee (\sim B \wedge \sim C))$$

which implies that

$$(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C).$$

□

AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA

3. The awful axioms.

$$\begin{aligned} A &\to (B \to A) \\ \sim A &\vee (\sim B \vee A) \\ \sim A &\vee A \vee B \\ \sim A &\vee A \end{aligned}$$

$$\begin{aligned} (A \to (B \to C)) &\to ((A \to B) \to (A \to C)) \\ \sim (\sim A \vee (\sim B \vee C)) &\vee (\sim (\sim A \vee B) \vee (\sim A \vee C)) \\ &(\sim\sim A \wedge \end{aligned}$$

4. Really neat!

Lemma 4.1. Suppose A, B, C are formulae and $A \leftrightarrow B$. Then

$$(A \Diamond C) \leftrightarrow (B \Diamond C)$$

if $\Diamond$ is one of $\vee, \wedge, \to, \leftrightarrow$.

Proof. It will be enough to show that

$$(A \Diamond C) \to (B \Diamond C)$$

if $A \leftrightarrow B$ and $\Diamond$ is one of $\vee, \wedge, \to, \leftrightarrow$. Since $A \leftrightarrow B$ we have

$$A \to B \quad \text{and} \quad B \to A.$$

Case One. $\Diamond = \vee$.

$$\begin{aligned} &A \vee C \\ &A \to B \\ &\sim A \vee B \\ &C \vee B \\ &B \vee C \end{aligned}$$

Case Two. $\Diamond = \wedge$.

$$\begin{aligned} &A \wedge C \\ &A \\ &C \\ &A \to B \\ &B \\ &B \wedge C \end{aligned}$$

Case Three. $\Diamond =\to$.

$$\begin{aligned} &A \to C \\ &B \to A \\ &B \to C \end{aligned}$$

Case Four. $\diamondsuit = \leftrightarrow$.

$$\begin{array}{ll} A \leftrightarrow C & \\ A \rightarrow C & \\ B \rightarrow C & \textbf{Case Three.} \\ C \rightarrow A & \\ C \rightarrow B & \textbf{Case Three.} \\ C \rightarrow A & \\ A \rightarrow B & \\ B \rightarrow C & \\ B \leftrightarrow C & \end{array}$$

□

Theorem 4.1. Suppose F is a formula, S is a formula F and S is a substring of T. Then S is a subformula of F.

Moreover, if T is a formula, $S \leftrightarrow T$ and G is the string obtained from F with S replaced by T then G is a formula and $F \leftrightarrow G$.

Proof. Let $\mathcal{S} = (\mathcal{M}, \mathbf{r}, P, \xi)$ and $\mathcal{F} = (\mathcal{N}, \mathbf{s}, \mathrm{o})$ be parse trees for S and F, respectively.

Part One. S is a subformula of F. Induct on the length L of S.

If $L = 1$ then S is a leaf node $\mathcal{T}$. Since S is a formula it must be a statement letter so S is a subformula of $\mathcal{T}$.

Suppose $L > 1$. Apply the inductive hypothesis to the branches of the children of $\mathbf{s}$ and conclude that the yield of each of these branches is a subformula of F; we may then conclude that S is a subformula of F.

Part Two. Suppose T is a formula, $S \leftrightarrow T$ and G is the string obtained from F with S replaced by T. Let $\mathcal{G}$ be the tree obtained by replacing the $\mathbf{s}$ branch of S by the parse tree for T. Clearly, $\mathcal{G}$ is a parse tree for G so G is a formula. Now one only has to note that $\sim S \leftrightarrow \sim T$ and $S \diamondsuit H \leftrightarrow T \diamondsuit H$ if $\diamondsuit$ is one of $\vee, \wedge, \rightarrow, \leftrightarrow$. □