1. Equivalence is preserved under substitution.

We fix a first order logic $\mathcal{F}$ and we let $\mathcal{S}$ be its set of statements.

Theorem 1.1. Suppose A, B and C_i, D_i, $i = 1, \ldots, N$ for some positive integer N satisfy the following conditions.

(i) $A \in \mathcal{S}$;
(ii) for each $i = 1, \ldots, N$, $C_i, D_i \in \mathcal{S}$ and $\vdash (C_i \leftrightarrow D_i)$;
(iii) for each $i = 1, \ldots, N$, C_i is a substring of A;
(iv) if $1 \le i < j \le N$ then the substrings C_i and C_j of A are disjoint;
(v) B is the string resulting from replacing each occurrence of C_i in A by D_i, $i = 1, \ldots, N$.

Then

$$B \in \mathcal{S} \quad \text{and} \quad \vdash (A \leftrightarrow B).$$

Proof. Well, it's a sketch. Let $\mathcal{T}$ be a parse tree for A.

Suppose $1 \le i \le N$. Consider the leaf node in $\mathcal{T}$ corresponding to the leftmost letter in C_i. Note that it is the leftmost member of the children of its parent. Thus the subtree of $\mathcal{T}$ corresponding to this parent must parse C_i. It follows that $B \in \mathcal{S}$.

To prove that $\vdash (A \leftrightarrow B)$ we induct on the depth of $\mathcal{T}$. Owing to previous theory we may assume without loss of generality that none of $\exists, \wedge, \rightarrow, \leftrightarrow$ occur in A. In case $A = \sim E$ for some $E \in \mathcal{S}$ or in case $A = (E \vee F)$ for some $E, F \in \mathcal{S}$ the assertion follows by earlier work (and is pretty clear regardless). So suppose $A = \forall\, x\, E$ for some $x \in X$ and some $E \in \mathcal{S}$. Then each C_i, $i = 1, \ldots, N$, is a substring of E so if F is the statement resulting by replacing each occurrence of C_i in F by D_i, $i = 1, \ldots, N$, we infer inductively that $\vdash (E \leftrightarrow F)$. To complete the proof we have only to note that $\{(E \rightarrow F)\} \vdash (\forall\, x\, E \rightarrow \forall\, x\, F)$ and $\{(F \rightarrow E)\} \vdash (\forall\, x\, F \rightarrow \forall\, x\, E)$ □