## 1. Model existence theorem.

We fix a first order logic $\mathcal{F}$ such that

$$C \neq \emptyset.$$

We let $\mathcal{S}$ be the set of statements of $\mathcal{F}$ and we suppose

$$\Gamma \subset \mathcal{S}.$$

We let

$$\mathbf{VFT}$$

be the set of variable free terms. For each $s \in \mathbf{VFT}$ we let

$$[s] = \{t \in \mathbf{VFT} : \Gamma \vdash (s = t)\}.$$

We have proved that

(i) $s \in [s]$;
(ii) $s \in [t]$ if $t \in [s]$;
(iii) $s \in [u]$ if $t \in [s]$ and $u \in [t]$.

That is, $\{(s,t) \in \mathbf{VFT} : t \in [s]\}$ is an equivalence relation on $bfVFT\}$ and $\{[s] : s \in \mathbf{VFT}\}$ is the set of equivalence classes.

We let

$$D = \{[s] : s \in \mathbf{VFT}\}.$$

We define

$$\mathbf{C} : C \to D$$

by letting $\mathbf{C})(c) = [c]$ for $c \in C$.

**Proposition 1.1.** There is one and only one function

$$\mathbf{F}$$

with domain $\mathbf{F}$ such that for each if $n \in \mathbb{N}^+$ and $f \in F_n$ then

$$\mathbf{F}(f) : D^n \to D$$

and

$$\mathbf{F}(f)([s_1], \ldots, [s_n]) = [f(s_1, \ldots, s_n)] \quad \text{whenever } s_1, s_2, \ldots, s_n \in \mathbf{VFT}.$$

There is one and only one function

$$\mathbf{R}$$

with domain $\mathbf{R}$ such that for each if $n \in \mathbb{N}^+$ and $r \in R_n$ then

$$\mathbf{R}(r) : D^n \to \{0, 1\}$$

and

$$\mathbf{R}(r)([s_1], \ldots, [s_n]) = 1 \Leftrightarrow \Gamma \vdash r(s_1, \ldots, s_n) \quad \text{whenever } s_1, s_2, \ldots, s_n \in \mathbf{VFT}.$$

*Proof.* This is a direct consequence of the Corollaries of the Equality Theorem. □

**Definition 1.1.** We call

$$\mathcal{I} = (D, \mathbf{C}, \mathbf{F}, \mathbf{R})$$

the **canonical interpretation of $\mathcal{F}$ with respect to $\Gamma$.**

**Proposition 1.2.** Suppose $\alpha \in D^X$ and $t \in \mathbf{VFT}$. Then

$$t_\alpha = [t] \quad \text{for } t \in \mathbf{VFT}.$$

*Proof.* The Proposition holds trivially if $t = c \in C$. Now induct on the depth of a parse tree for $t$. □

**Corollary 1.1.** Suppose $s, t \in \mathbf{VFT}$. Then

$$\Gamma \vdash (s = t) \;\Leftrightarrow\; (s = t) \text{ is true in } \mathcal{I}.$$

*Proof.* One need only observe that

$$\mathbf{t}_{(s=t)}(\alpha) = 1 \;\Leftrightarrow\; s_\alpha = t_\alpha \;\Leftrightarrow\; [s] = [t] \;\Leftrightarrow\; \Gamma \vdash (s = t)$$

for any $\alpha \in D^X$. □

**Corollary 1.2.** Suppose $n \in \mathbb{N}^+$, $r \in R_n$ and $s_1, \dots, s_n \in \mathbf{VFT}$. Then

$$\Gamma \vdash r(s_1, \dots, s_n) \;\Leftrightarrow\; r(s_1, \dots, s_n) \text{ is true in } \mathcal{I}.$$

*Proof.* One need only observe that

$$\begin{aligned} \mathbf{t}_{r(s_1,\dots,s_n)}(\alpha) = 1 & \\ &\Leftrightarrow \mathbf{R}((s_1)_\alpha, \dots, (s_n)_\alpha) = 1 \\ &\Leftrightarrow \mathbf{R}([s_1], \dots, [s_n]) = 1 \\ &\Leftrightarrow \Gamma \vdash r(s_1, \dots, s_n) \end{aligned}$$

for any $\alpha \in D^X$. □

**Proposition 1.3.** Suppose $A$ is a statement. Then there is a statement $B$ which contains no occurrence of $\exists, \wedge, \rightarrow, \leftrightarrow$ such that

$$\vdash (A \leftrightarrow B).$$

*Proof.* Let $\mathcal{B}$ be the set of statements $B$ which contain no occurrence of $\exists, \wedge, \rightarrow, \leftrightarrow$.

We induct on the depth of a parse tree for $A$. If $A \in \mathcal{B}$ we can take $B = A$ since

$$(\sim A \vee A)$$
$$(A \rightarrow A)$$
$$(A \rightarrow A) \wedge (A \rightarrow A)$$
$$(A \leftrightarrow A)$$

is a proof of $(A \leftrightarrow A)$. In particular, this will be the case of $A$ is atomic.

Suppose $A = \sim C$ for some $C \in \mathcal{S}$. By induction, there is $D \in \mathcal{B}$ such that $\vdash (C \leftrightarrow D)$ Then $B = \sim D \in \mathcal{B}$ and

$$(C \leftrightarrow D)$$
$$(\sim C \leftrightarrow \sim D)$$

is a proof of $(A \leftrightarrow B)$.

Suppose $A = \exists x\, C$ for some $x \in X$ and $C \in \mathcal{S}$. By induction there is $D \in \mathcal{B}$ such that $\vdash (C \leftrightarrow D)$. Let $B = \sim \forall x\, D$. Then $B \in \mathcal{B}$

$$(\exists x\, C \leftrightarrow \sim \forall x\, \sim C)$$
$$(C \leftrightarrow D)$$
$$(\sim C \leftrightarrow \sim D)$$
$$(\forall x\, \sim C \leftrightarrow \forall x\, \sim D)$$
$$(\sim \forall x\, \sim C \leftrightarrow \sim \forall x\, \sim D)$$

is a proof of $(A \leftrightarrow B)$.

Suppose $A = (C\, o\, D)$ where $C, D \in \mathcal{S}$ and $o \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$. By induction there are $E, F \in \mathcal{B}$ such that $\vdash (C \leftrightarrow E)$ and $\vdash (D \leftrightarrow F)$. Let

$$B = \begin{cases} (E \vee F) & \text{if } o = \vee, \\ \sim (\sim E \vee \sim F) & \text{if } o = \wedge, \\ (\sim E \vee F) & \text{if } o = \rightarrow, \\ \sim (\sim (\sim E \vee F) \vee \sim (\sim F \vee E)) & \text{if } o = \leftrightarrow. \end{cases}$$

Then $B \in \mathcal{B}$ and $\vdash (A \leftrightarrow B)$. □

**Definition 1.2.** We say $\Gamma$ is **Henkin** if for each $A \in \mathcal{S}$ such that $\mathbf{free}(A) = \{x\}$ for some $x \in X$ and

$$\Gamma \vdash \sim \forall x\, A$$

there is $t \in \mathbf{VFT}$ such that

$$\Gamma \vdash \sim A_{x \rightarrow t}.$$

(Note that $t \in \mathbf{subs}(x, A)$.) One calls such a $t$ a **witness to** $\sim \forall x\, A$**.**

**Theorem 1.1. (Henkin)** Suppose $\Gamma$ is consistent, complete and Henkin. Then $\mathcal{I}$ is a model for $\Gamma$.

*Proof.* Let

$$\begin{aligned} \mathcal{A} &= \{A \in \mathcal{S} : \Gamma \vdash A \ \text{ and } A \text{ is true in } \mathcal{I}\}; \\ \mathcal{B} &= \{A \in \mathcal{S} : \Gamma \vdash A \ \text{ and } A \text{ is false in } \mathcal{I}\}; \\ \mathcal{C} &= \{A \in \mathcal{S} : \Gamma \vdash \sim A \ \text{ and } A \text{ is true in } \mathcal{I}\}; \\ \mathcal{D} &= \{A \in \mathcal{S} : \Gamma \vdash \sim A \ \text{ and } A \text{ is false in } \mathcal{I}\}. \end{aligned}$$

Since $\Gamma$ is complete and consistent we find that for any $A \in \mathcal{S}$ exactly one of

$$\Gamma \vdash A \quad \text{or} \quad \Gamma \vdash \sim A \tag{1}$$

holds. Also, if $A \in \mathcal{S}$ and $A$ is a sentence exactly one of

$$A \text{ is true in } \mathcal{I} \quad \text{or} \quad A \text{ is false in } \mathcal{I} \tag{2}$$

holds; this is because if $\alpha, \beta \in D^X$ then $\mathbf{t}_A(\alpha) = \mathbf{t}_B(\beta)$ since $\alpha$ and $\beta$ agree on $\mathbf{free}(A) = \emptyset$. Thus if $A$ is a sentence then $A$ belongs to exactly one of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ or $\mathcal{D}$.

**Lemma 1.1.** Suppose $A$ is a sentence. Then $A \in \mathcal{A} \cup \mathcal{D}$.

*Proof.* By virtue of the preceding Proposition we may assume that $A$ has no occurrence of $\exists, \wedge, \rightarrow$ or $\leftrightarrow$. We induct on the number of occurrences of $\sim, \vee$ and $\forall$ in $A$.

**Part One.** Suppose $A$ is atomic. A preceding Proposition implies $A \in \mathcal{A} \cup \mathcal{D}$.

**Part Two.** Suppose $B \in \mathcal{S}$ and $A = \sim B$. Then $B$ is a sentence and so, by induction, $B \in \mathcal{A} \cup \mathcal{D}$. In case $B \in \mathcal{A}$ we find that $A \in \mathcal{D}$ and in case $B \in \mathcal{D}$ we find that $A \in \mathcal{A}$.

**Part Three.** Suppose $A = (B \vee C)$. Then $B$ and $C$ are sentences and so, by induction, $B$ and $C$ belong to $\mathcal{A} \cup \mathcal{D}$. In case $B \in \mathcal{A}$ and $C \in \mathcal{A}$ or $B \in \mathcal{A}$ and $C \in \mathcal{D}$ of $B \in \mathcal{D}$ and $C \in \mathcal{A}$ we find that $A \in \mathcal{A}$. In case $B \in \mathcal{D}$ and $C \in \mathcal{D}$ we find that $A \in \mathcal{D}$.

**Part Four.** Suppose $A = \forall x\, B$ for some $x \in X$ and some statement $B$ such that $x \notin \mathbf{free}(B)$. Then $B$ is a sentence and, by induction, $B \in \mathcal{A} \cup \mathcal{D}$. Since $B$ is a sentence we have $\mathbf{t}_A = \mathbf{t}_B$.

In case $B \in \mathcal{A}$ we have $\Gamma \vdash A$ by the closure theorem for provability (p. 172) so $A \in \mathcal{A}$.

In case $B \in \mathcal{D}$ we have $\vdash\sim A$ since if it were the case that $\Gamma \vdash A$ we would have $\Gamma \vdash B$ by the closure theorem for provability. Thus $A \in \mathcal{D}$.

**Part Five.** Suppose $A = \forall x\, B$ for some $x \in X$ and some statement $B$ such that $x \in \mathbf{free}(B)$.

Suppose $\Gamma \vdash A$. Let $\alpha \in D^X$; we need to show that $\mathbf{t}_A(\alpha) = 1$. So suppose $\beta \in D^X$ and $\beta \sim_x \alpha$. Let $t \in \mathbf{VFT}$ be such that $\alpha(x) = [t]$. Then $t_\alpha = x_\alpha$ so $t_\alpha = x_\alpha = x_\beta$ which, by a theorem we have already proved on substitution, implies $\mathbf{t}_{B_{x\to t}}(\alpha) = \mathbf{t}_B(\beta)$. Moreover,

$$\forall x\, B$$
$$(\forall x\, B \to B_{x\to t})$$
$$B_{x\to t}$$

gives $\Gamma \vdash B_{x\to t}$. Now $B_{x\to t}$ is a sentence to which the inductive hypothesis applies so $B_{x\to t}$ is true in $\mathcal{I}$ and, therefore, $\mathbf{t}_{B_{x\to t}}(\alpha) = 1$. Thus $\mathbf{t}_B(\beta) = 1$ so $A \in \mathcal{A}$, as desired.

Suppose $\Gamma \vdash\sim A$ which is to say that $\vdash\sim \forall x\, B$. Since $\Gamma$ is Henkin there is $t \in \mathbf{VFT}$ such that $\Gamma \vdash\sim B_{x\to t}$. Now $B_{x\to t}$ is a sentence to which the inductive hypothesis applies so $B_{x\to t}$ is false in $\mathcal{I}$. The substitution axiom $(\forall x\, B \to B_{x\to t})$ implies $\sim A$ is false in $\mathcal{I}$. Thus $A \in \mathcal{D}$.

□

Now suppose $\Gamma \vdash A$. Then $\Gamma \vdash A'$ where $A'$ is a closure of $A$. By the Lemma, $A'$ is true in $\mathcal{I}$. By the Closure Theorem for Interpretations (p. 152), $A$ is true in $\mathcal{I}$. □

**Theorem 1.2. (Theorem on constants, p. 194)** Suppose $A \in \mathcal{S}$; $\Gamma \vdash A$; $c \in C$ is such that $c$ does not occur in $A$ or in any statement of $\Gamma$; and $x \in X$ is such that $\Gamma \vdash A_{x\to c}$. Then $A \vdash \forall x\, A$.

*Proof.* Suppose $A_1, \ldots, A_n$ is a primary proof of $A_{x\to c}$ using $\Gamma$. Let $y \in X$ be such that $y$ does not occur in $A_1, \ldots, A_n$ or $A$. Let $B_i$, $i = 1, \ldots, n$ be obtained by replacing each occurence of $c$ in $A_i$ with $y$. Observe that $B_i \in \mathcal{S}$ for $i = 1, \ldots, n$.

I claim that $B_1, \ldots, B_n$ is a proof of $A_{x\to y}$. First, observe that $B_n = A_{x\to y}$. Next observe that $B_i = A_i$ if $A_i \in \Gamma$. We leave it as an exercise for the reader to verify that if $j \in \{1, \ldots, n\}$, $I \subset \{i \in \{1, \ldots, n\} : i < j\}$ and $(\{A_i : i \in I\}, A_j)$ is a rule of inference then so is $(\{B_i : i \in I\}, B_j)$; it will be necessary to use the hypothesis that $c$ does not occur in $A$ or in any $A_1, \ldots, A_n$. Thus $\Gamma \vdash A_{x\to y}$.

Appending $\forall y\, A_{x\to y}, (\forall y\, A_{x\to y} \to A), A, \forall x\, A$ to $B_1, \ldots, B_n$ is a proof of $\forall x\, A$ using $\Gamma$; □

**Corollary 1.3.** Suppose $\Gamma$ is consistent, $A \in \mathcal{S}$, $x \in X$, $\mathbf{free}(A) = \{x\}$ and $\Gamma \vdash\sim \forall x\, A$. Suppose $c \in C$ and $c$ does not occur in $A$ or in any statement of $\Gamma$. Then $\Gamma \cup \{\sim A_{x\to c}\}$ is consistent.

*Proof.* Suppose, contrary to the Corollary, $\Gamma \cup \{\sim A_{x\to c}\}$ is not consistent. Then $\Gamma \cup \{\sim A_{x\to c}\} \vdash A_{x\to c}$. By the Deduction Theorem, $\Gamma \vdash (\sim A_{x\to c} \to A_{x\to c})$. This implies $\Gamma \vdash A_{x\to c}$. The Theorem on Constants now gives $\Gamma \vdash \forall x\, A$ which contradicts the consistency of $\Gamma$. □

**Theorem 1.3. (Lindenbaum-Henkin)** Suppose $\Gamma$ is consistent and $c$ is a univalent sequence in $C$ such that no member of the range of $c$ occurs in any statement of $\Gamma$. Suppose also that $C, X, F, R$ are countable. Then there is a set of formulas $\Delta$ such that $\Gamma \subset \Delta$ and $\Delta$ is consistent, complete and Henkin.

*Proof.* We let $\mathcal{A}$ be the set of $(\Lambda, A)$ such that $\Lambda \subset \mathcal{S}$, $A \in \mathcal{S}$,

$$\Lambda \vdash A \text{ and } A = \forall\, x\, B \text{ for no } (x, B) \in X \times \mathcal{S} \text{ with } \mathbf{free}(B) = \{x\};$$

we let $\mathcal{B}$ be the set of $(\Lambda, A)$ such that $\Lambda \subset \mathcal{S}$, $A \in \mathcal{S}$,

$$\Lambda \vdash A \text{ and } A = \forall\, x\, B \text{ for some } (x, B) \in X \times \mathcal{S} \text{ with } \mathbf{free}(B) = \{x\};$$

we let $\mathcal{C}$ be the set of $(\Lambda, A)$ such that

$$\text{it is not the case that } \Lambda \vdash A.$$

Note that the set of sentences is countably infinite. Let $A$ be an enumeration of the set of sentences.

We construct a sequence $\Gamma_n$, $n \in \mathbb{N}$ in $\mathcal{S}$ and a sequence $N$ in $\mathbb{N}$ inductively as follows. We let $\Gamma_0 = \Gamma$ and we let $N_0 = 0$. For each $n \in \mathbb{N}$ we require that

$$N_{n+1} = \begin{cases} N_n & \text{if } (\Gamma_n, A_n) \in \mathcal{A}; \\ N_n + 1 & \text{if } (\Gamma_n \vdash A_n) \in \mathcal{B}; \\ N_n & \text{if } (\Gamma_n, A_n) \in \mathcal{C}. \end{cases}$$

and that

$$\Gamma_{n+1} = \begin{cases} \Gamma_n & \text{if } (\Gamma_n, A_n) \in \mathcal{A}; \\ \Gamma_n \cup \{\sim A_{x \to c_{N_n}}\} & \text{if } (\Gamma_n, A_n) \in \mathcal{B}; \\ \Gamma_n \cup \{\sim A_n\} & \text{if } (\Gamma_n, A_n) \in \mathcal{C}. \end{cases}$$

See page 196 for the remaining details of the proof. □

For the remainder of this section let us suppose $C \subset C'$ and that $\mathcal{F}'$ is the first order logic obtained from $\mathcal{F}$ by replacing $C$ with $C'$.

**Theorem 1.4. (Extension by constants. p. 196)** For each $A \in \mathcal{S}$ we have

$$\Gamma \vdash_{\mathcal{F}} A \;\Leftrightarrow\; \Gamma \vdash_{\mathcal{F}'} A.$$

Moreover,

$$\Gamma \text{ is consistent with respect to } \mathcal{F} \;\Leftrightarrow\; \Gamma \text{ is consistent with respect to } \mathcal{F}'.$$

*Proof.* See page 196. □

**Theorem 1.5. Model existence theorem.** Suppose $\Gamma$ is consistent. Then $\Gamma$ has a model.

*Proof.* See page 197. □