1. n-TUPLES.

We let

$$\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$$

be the set of **natural numbers** and we let

$$\mathbb{N}^+ = \{n \in \mathbb{N} : n > 0\}.$$

For each $n \in \mathbb{N}$ we let

$$I(n) = \{m \in \mathbb{N} : m < n\};$$

Note that $I(n)$ has n members and that $I(0) = \emptyset$.

If A and B are sets we say A **is equipollent with** B and write

$$A \approx B$$

if there are $f : A \to B$ such that $f^{-1} : B \to A$. Note that the relation $\approx$ is reflexive, symmetric and transitive. Whenever A is a set we let

$$|A| = \begin{cases} n & \text{if } n \in \mathbb{N} \text{ and } A \approx I(n), \\ \infty & A \approx I(n) \text{ for no } n \in \mathbb{N}. \end{cases}$$

Suppose $n \in \mathbb{N}$. An n-**tuple** is, by definition, a function whose domain is $I(n)$. Thus the only 0-tuple is the empty function which equals the empty set. Note that if x is an n-tuple then $|x| = n$; we call $|x|$ the **length of** x. We say x is **tuple** if x is an n-tuple for some $n \in \mathbb{N}$.

Suppose $n \in \mathbb{N}^+$ and $b_0, b_1, \ldots, b_{n-1}$ are objects. Let $a = \{(i, b_i) : i \in I(n)\}$ and note that a is an n-tuple such that $a_i = b_i$, $i \in I(n)$. We will each of the following three notations

$$(b_0, b_1, \ldots, b_{n-1})$$

$$b_0 \ b_1 \ \cdots \ b_{n-1}$$

and

$$b_0 b_1 \cdots b_{n-1}$$

to denote a; each has its problems. Among other things, this notation is ambiguous because if x, y are objects we also use (x, y) to denote the ordered pair with first coordinate x and second coordinate y.

Suppose x is a n-tuple, $i, l \in \mathbb{N}$ and $i + l - 1 \in I(n)$. We let

$$\mathbf{s}_{i,l}(x) = \begin{cases} \emptyset & \text{if } l = 0, \\ \{(j, x_{i+j}) : j \in I(l)\} & \text{if } l > 0; \end{cases}$$

Note that $\mathbf{s}_{i,l}(x)$ is an l-tuple.

We say the tuple x is a **subtuple** of the tuple y if $|x| \le |y|$ and $x = \mathbf{s}_{i,l}(y)$ for some $i, l \in \mathbb{N}$ with $i + l - l \in I(|y|)$.

Suppose x and y are tuples. We let

$$x \wedge y = \bigcup_{n=0}^{\infty} \{x|I(n) : x|I(n) = y|I(n)\}$$

and note that $x \wedge y$ is a subtuple of both x and y and that $|x \wedge y| \le \min\{|x|, |y|\}$. Note that the $\wedge$ operation is commutative and associative.

Suppose x is a n-tuple, $i, l \in \mathbb{N}$, $i + l - l \in I(|x|)$ and z is some object. We let

$$\mathbf{m}_{i,l}(x, z) = \{k \in I(l) : s_{i+l} = z\}.$$

If A is a set we let

$$A^n$$

be the set of n-tuples whose range is a subset of A.

1.1. **Concatenation.** Given an m-tuple s and an n-tuple t we let their **concatenation** $s|t$ be the $m+n$-tuple

$$s \cup \{(m+i, t(i) : i \in I(n)\}$$

and note that

$$\mathbf{rng}\, s|t = \mathbf{rng}\, s \cup \mathbf{rng}\, t.$$

Note that the operation of concatenation is associative but not commutative.

2. Alphabets and languages.

Let A be a nonempty set which we call the **alphabet**. A member of $\bigcup_{n=0}^{\infty} A^n$ is called a **word** (or sometimes a **string** or **sentence** or **formula** or who knows what else...). It is customary to use ϵ instead of $\emptyset \in A^0$ to denote the empty word.

We say L is a **language (on the alphabet** A**)** if $L \subset \bigcup_{n=0}^{\infty} A^n$. We let

$$(A) = \{(a) : a \in A\}$$

and note that (A) is a language on the alphabet A.

Suppose L and M are languages over the alphabets A and B, respectively. Let

$$L|M = \{s|t : s \in L \text{ and } t \in M\}$$

and note that LM is a language over $A \cup B$; note that this operation is associative. For each $n \in \mathbb{N}$ we define

$$L^{(n)}$$

inductively as follows: $L^{(0)} = \{\epsilon\}$ and $L^{(n+1)} = (L^{(n)})L$. We let

$$L^* = \bigcup_{n=0}^{\infty} L^{(n)}$$

and call this language over A the **Kleene closure of** L. We let

$$L^+ = \bigcup_{n=0}^{\infty} L^{(n)}$$

and call this language over A the **positive closure of** L.

Note that

$$(A)^* = \bigcup_{n=0}^{\infty} A^n.$$

Suppose $<$ well orders A. We define a well ordering on $(A)^*$, which also denote by $<$, called the **lexicographical ordering (induced by** $<$**)**, as follows. Suppose $a, b \in (A)^*$. Then $a < b$ if there is $j \in \mathbb{N}$ such that $a|I(j) = b|I(j)$ and *either* $|a| = j < |b|$ *or* $|a| > j$, $|b| > j$ and $a_j < b_j$.

2.1. **Word forms.** Suppose L is a language over the alphabet A. A **word form for** L is a member S of $(A \cup B)^*$ where B is disjoint from A with the property that whenever

$$\alpha : B \to L$$

then

$$\hat{\alpha}(S) \in L$$

where $\hat{\alpha}(S)$ is that member of $(A)^*$ obtained by replacing each occurrence of a member of B in S by $\alpha(B)$.

3. Formal theories..

Definition 3.1. By a **formal theory** we mean an ordered triple

$$\mathcal{T} = (L, A, \mathcal{R})$$

such that L is a language on the alphabet A and $\mathcal{R}$ is a family of ordered pairs (H, s) such that H is a finite subset of L and $s \in L$. In this context a member of L is frequently called a **statement** and a member of $\mathcal{R}$ is called **rules of inference**. If (H, s) is a rule of inference and H is empty we say s is an **axiom**.

If $\Gamma \subset L$ and $s \in L$ a **proof of** s **using** Γ is a tuple P such that $n+1 = |P| > 0$, $P_n = s$ and such that for each $j \in I(n+1)$ *either* $P_j \in \Gamma$ *or* there is a subset H of $\{P_i : 0 \le i < j\}$ such that (H, P_j) is a rule of inference; one writes

$$\Gamma \vdash s$$

and says s **is a theorem of** Γ. If Γ is empty we write

$$\vdash s$$

in which case we say s is a **theorem**;

This definition may be more general than what you will find in many books but it is nonetheless useful and it has the advantage, unlike many you see in books, of being precise.

Definition 3.2. Suppose $\mathcal{T} = (L, A, \mathcal{R})$ is a formal theory and $\Gamma \subset L$ and

$$\sim$$

is a member of the alphabet A such that

$$\sim s$$

is in the language L whenever s is in the language. (In what follows $\sim$ will always be negation.)

$$\mathbf{t}(\Gamma) = \{s \in L : s \text{ is a theorem of } \Gamma\};$$
$$\mathbf{i}(\Gamma) = \{s \in L : \text{both } s \text{ and } \sim s \text{ are theorems of } \Gamma\};$$
$$\mathbf{c}(\Gamma) = \{s \in L : \text{either } s \text{ or } \sim s \text{ are theorems of } \Gamma\}.$$

We say Γ is **consistent (with respect to $\sim$)** if $\mathbf{i}(\Gamma)$ is empty. We say Γ is **complete (with respect to $\sim$)** if $\mathbf{c}(\Gamma) = L$.

We say Γ is **decidable** if for any $s \in L$ there is an algorithm for deciding whether or not $s \in \mathbf{t}(\Gamma)$. Of course this doesn't mean much until we say what "there is an algorithm for..." means.